TWO FORMULAS RELATED TO TWO-DIMENSIONAL NORMAL DISTRIBUTION

SEPPO MUSTONEN

ABSTRACT. Generalized Box-Müller formulas for generating two-dimensional normal distribution with correlation coefficient ρ are derived. Also equations for contour ellipses of the same distribution are derived in a parametric form suitable for plotting. Both results were presented in the lecture notes "Statistical multivariate methods" by the author in 1995 (in Finnish).

1. GENERALIZATION OF THE BOX-MÜLLER FORMULAS

Let U_1 and U_2 be independent variables from the uniform distribution on $(0, 1)$. According to the well-known Box-Müller formulas

$$\begin{aligned} V_1 &= \sqrt{-2\log U_2}\cos(2\pi U_1), \\ V_2 &= \sqrt{-2\log U_2}\sin(2\pi U_1). \end{aligned}$$

V_1 and V_2 are independent N(0,1) variables.

By the linear transformation

$$\begin{aligned} W_1 &= V_1, \\ W_2 &= \rho V_1 + \sqrt{1-\rho^2}\, V_2 \end{aligned}$$

two standard normal variables W_1 and W_2 with a correlation coefficient ρ are obtained.

According to the definition of the V variables we have

$$\begin{aligned} W_1 &= \sqrt{-2\log U_2}\cos(2\pi U_1), \\ W_2 &= \sqrt{-2\log U_2}\,[\rho\cos(2\pi U_1) + \sqrt{1-\rho^2}\sin(2\pi U_1)] \\ &= \sqrt{-2\log U_2}\,[\sin(\arcsin(\rho))\cos(2\pi U_1) + \cos(\arccos(\rho))\sin(2\pi U_1)] \\ &= \sqrt{-2\log U_2}\ \sin(2\pi U_1 + \arcsin(\rho)). \end{aligned}$$

Then a two-dimensional normal variable (X_1, X_2) with expected values (μ_1, μ_2), standard deviations (σ_1, σ_2) and correlation ρ is generated by

$$\begin{aligned} X_1 &= \mu_1 + \sigma_1\sqrt{-2\log(U_2)}\cos(2\pi U_1), \\ X_2 &= \mu_2 + \sigma_2\sqrt{-2\log(U_2)}\sin(2\pi U_1 + \arcsin(\rho)). \end{aligned}$$

These Mustonen formulas were also published in "Mathematics Handbook" by Råde and Westergren in 1995 (p.425), Studentlitteratur, Lund.

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2. Contour curves for the two-dimensional normal distribution

It is well known that a multivariate normal distribution $\mathbf{X} \sim N(\mu, \mathbf{\Sigma})$ has a contour ellipse at confidence level P with an equation

$$(\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu) \sim \chi_p^2(P). \tag{1}$$

In the two-dimensional case this can be represented in a parametric form

$$\begin{aligned} X_1 &= \mu_1 + \sigma_1\sqrt{-2\log(1-P)}\cos(t), \\ X_2 &= \mu_2 + \sigma_2\sqrt{-2\log(1-P)}\sin(t+\arcsin(\rho)) \end{aligned} \tag{2}$$

for $0 \le t \le 2\pi$.

It is sufficient to prove (2) in the case of expected values 0 and unit variances. Then the contour ellipse takes the form

$$\frac{1}{1-\rho^2}(x_1^2 - 2\rho x_1 x_2 + x_2^2) = \chi_2^2(P) = -2\log(1-P). \tag{3}$$

By denoting $C = \sqrt{-2\log(1-P)}$ equations (2) are reduced into the form

$$\begin{aligned} x_1 &= C\cos(t), \\ x_2 &= C\sin(t+\arcsin(\rho)). \end{aligned} \tag{4}$$

Now it is shown that eliminating t from (4) leads to (3).
By using $\cos(t) = x_1/C$ we obtain

$$\begin{aligned} x_2 &= C(\sin(t)\cos(\arcsin(\rho)) + \cos(t)\sin(\arcsin(\rho))) \\ &= C(\sin(t)\sqrt{1-\rho^2} + \cos(t)\rho) \\ &= C(\sqrt{(1-x_1^2/C^2)(1-\rho^2)} + (x_1/C)\rho). \end{aligned}$$

By moving $x_1\rho$ to the left side and by squaring, this equation simplifies to

$$x_1^2 - 2\rho x_1 x_2 + x_2^2 = C^2(1-\rho^2)$$

which is identical with (3). □

Department of Mathematics and Statistics, University of Helsinki
E-mail address: `seppo.mustonen@helsinki.fi`