ON CERTAIN CROSS SUM PUZZLES

SEPPO MUSTONEN
2 JUNE 2006

A class of cross sum puzzles called Survo puzzles will be studied. Survo puzzles are somewhat related to Kakuro cross sum puzzles but played on a simple, open rectangular grid and without limiting to integers from 1 to 9.

In a Survo puzzle the task is to fill an $m \times n$ –table by integers $1, 2, \ldots, mn$ so that each of these numbers appears only once and their row and column sums are equal to integers given on the bottom and the right side of the table. Often some of the integers are given readily in the table in order to guarantee uniqueness of the solution and/or for making the task easier.

During last months I have presented various examples of Survo puzzles ("Survo-ristikoita" in Finnish) for active Survo users in the newsgroup (www.survo.fi ⇒ Keskustelu). In harder cases various computational features of the Survo system have been helpful.

1. Example 1

We are studying a 3×4 –puzzle

	6			30
8				18
		3		30
27	16	10	25	

For example, the sum of the numbers in the first row should be 30 and so the sum of the three missing numbers is 24. Similarly, the sum of the second column is 16 and hence the sum of two missing numbers is 10.

There are altogether 9 missing numbers and they are 1,2,4,5,7,9,10,11,12. Solution starts typically from a row or column with the least number of missing values. In this table columns 1,2, and 3 are such ones. Column 1 is not favorable since the sum 19 of missing numbers can be presented according to the rules in several ways (e.g. $19 = 7 + 12 = 12 + 7 = 9 + 10 = 10 + 9$). In the column 2 the sum of missing numbers is 10 having only one partition $10 = 1 + 9$ since the other alternatives $10 = 2 + 8 = 3 + 7 = 4 + 6$ are not accepted due to numbers already present in the table.

 seppo.mustonen@survo.fi
This is a revised translation from the original Finnish version. I am especially grateful for *Kimmo Vehkalahti* and *Reijo Sund* for their encouragement during preparation of this paper.

Number 9 cannot be put in the row 2 since then the sum of this row would exceed the value 18. Therefore the only choice is to start the solution by

	6			30
8	1			18
	9	3		30
27	16	10	25	

Now the first column has only one alternative $27 - 8 = 19 = 7 + 12 = 12 + 7$. Number 7 cannot be in the row 1 because the sum of missing numbers in that row would be $30 - 7 - 6 = 17$ and this allows no permitted partition. Thus we have

12	6			30
8	1			18
7	9	3		30
27	16	10	25	

implying that the last number in the last row will be $30 - 7 - 9 - 3 = 11$:

12	6			30
8	1			18
7	9	3	11	30
27	16	10	25	

In the first row the sum of the missing numbers is $30 - 12 - 6 = 12$. Its only possible partition is $12 = 2 + 10$ and so that number 2 will be in column 3; 10 in this position is too much for the column sum.

12	6	2	10	30
8	1			18
7	9	3	11	30
27	16	10	25	

The solution is then easily completed by

12	6	2	10	30
8	1	5	4	18
7	9	3	11	30
27	16	10	25	

2. Example 2

		7					8			55
	13								16	155
22	18	22	30	12	24	18	25	14	25	

This 2×10 –table is also easy after detecting that the first row sum 55 can be partitioned only as $1+2+3+4+5+6+7+8+9+10 = 55$. It is immediately updated into form

	5	7					8		9	55
	13	15					17		16	155
22	18	22	30	12	24	18	25	14	25	

Since all numbers in row 1 are at most 10, the sum 30 of the fourth column is presented uniquely as $30 = 10 + 20$ and we get

	5	7	10				8		9	55
	13	15	20				17		16	155
22	18	22	30	12	24	18	25	14	25	

Thereafter the solution is found by simple steps, for example, as follows:

	5	7	10	1			8		9	55
	13	15	20	11			17		16	155
22	18	22	30	12	24	18	25	14	25	

	5	7	10	1	6		8		9	55
	13	15	20	11	18		17		16	155
22	18	22	30	12	24	18	25	14	25	

3	5	7	10	1	6		8		9	55
19	13	15	20	11	18		17		16	155
22	18	22	30	12	24	18	25	14	25	

3	5	7	10	1	6	4	8		9	55
19	13	15	20	11	18	14	17		16	155
22	18	22	30	12	24	18	25	14	25	

3	5	7	10	1	6	4	8	2	9	55
19	13	15	20	11	18	14	17	12	16	155
22	18	22	30	12	24	18	25	14	25	

3. Example 3

Simple Survo puzzles like those already presented can be solved by basic arithmetics and reasoning.

As an exhibit how Survo – and especially its `COMB` program – can be used when dealing with harder puzzles let's study the next 5×5 table where rows are notated by 1–5 and columns by letters A–E, for easier reference.

	A	B	C	D	E	
1				2	3	35
2	14	4			12	57
3	8	6		13		65
4	5					83
5		9	25		11	85
	62	35	77	81	70	

The best starting point is maybe trying to detect which are the three missing numbers in row 1. We have to find out all the combinations of three distinct integers giving the sum $35 - 2 - 3 = 30$ with the restriction that numbers already present in the table must not be used. All possible alternatives are found by the following

COMB scheme. The integers already consumed are told to the COMB command by a specification OFF=2,3,14,4,12,8,6,13,5,9,25,11 and all partitions of 30 into a sum of 3 distinct (DISTINCT=1) integers are then obtained by the command

```
-------------------------------------------------------------------
OFF=2,3,14,4,12,8,6,13,5,9,25,11
COMB P,CUR+1 / P=PARTITIONS,30,3  DISTINCT=1
Partitions 3 of 30: N[P]=2
1 7 22
1 10 19
-------------------------------------------------------------------
```

COMB gives two alternative partitions $30 = 1 + 7 + 22 = 1 + 10 + 19$. Now COMB is applied to column B:

```
-------------------------------------------------------------------
35-4-6-9=16 is the sum of missing numbers.
COMB P,CUR+1 / P=PARTITIONS,16,2  DISTINCT=1
Partitions 2 of 16: N[P]=1
1 15
-------------------------------------------------------------------
```

Since number 15 cannot appear in row 1 (according to what is already known about this row) we must have B1=1 and B4=15:

	A	B	C	D	E	
1		1		2	3	35
2	14	4			12	57
3	8	6		13		65
4	5	15				83
5		9	25		11	85
	62	35	77	81	70	

Column A is studied next:

```
-------------------------------------------------------------------
62-14-8-5=35
OFF=2,3,14,4,12,8,6,13,5,9,25,11,1,15 (also 1 and 15 included)
COMB P,CUR+1 / P=PARTITIONS,35,2  DISTINCT=1  MAX=24
Partitions 2 of 35: N[P]=2
16 19
17 18
-------------------------------------------------------------------
```

Since neither 17 nor 18 can appear in row 1, the correct partitioning of row 1 is 19+1+10 implying A1=19, C1=10 and also A5=16.

These conclusions lead to

	A	B	C	D	E	
1	19	1	10	2	3	35
2	14	4			12	57
3	8	6		13		65
4	5	15				83
5	16	9	25		11	85
	62	35	77	81	70	

showing that D5 is $85 - 16 - 9 - 25 - 11 = 24$

	A	B	C	D	E	
1	19	1	10	2	3	35
2	14	4			12	57
3	8	6		13		65
4	5	15				83
5	16	9	25	24	11	85
	62	35	77	81	70	

Let's study row 2 and column D:

```
--------------------------------------------------------------------
Row 2:
57-14-4-12=27
OFF=2,3,14,4,12,8,6,13,5,9,25,11,1,15,19,10,16,24
COMB P,CUR+1 / P=PARTITIONS,27,2  DISTINCT=1  MAX=23
Partitions 2 of 27: N[P]=1
7 20

Column D:
81-2-13-24=42
COMB P,CUR+1 / P=PARTITIONS,42,2  DISTINCT=1  MAX=23
Partitions 2 of 42: N[P]=1
20 22
--------------------------------------------------------------------
```

Then D2=20 and furthermore C2=7, D4=22:

	A	B	C	D	E	
1	19	1	10	2	3	35
2	14	4	7	20	12	57
3	8	6		13		65
4	5	15		22		83
5	16	9	25	24	11	85
	62	35	77	81	70	

Only 4 numbers are missing and they are 17,18,21,23. The final solution is obtained easily:

	A	B	C	D	E	
1	19	1	10	2	3	35
2	14	4	7	20	12	57
3	8	6	17	13	21	65
4	5	15	18	22	23	83
5	16	9	25	24	11	85
	62	35	77	81	70	

4. Example 4

Still trickier is the next Survo puzzle but even this one can be solved without the aid of a computer as shown here by *Olli Mustonen* (a direct quotation of his message):

	A	B	C	D	E	
1	24					60
2	19			11	13	88
3		21				40
4			14	6		70
5		12			17	67
	64	84	56	36	85	

I noticed that in row 2 number 45 is needed as a sum of two integers. When using numbers available this is possible in two ways only: 20+25 and 22+23. Large numbers are required also in column E – the cell E3 is problematic because it cannot hold a large number since the sum of 4 missing numbers in row 3 can be only 19. After a short study I concluded that E3 must be one of the numbers 7, 8, 10. The sum of the remaining two numbers in column E can then be 45, 47, or 48 – these sums can be composed only from numbers 20, 22, 23, and 25 in different combinations. Since two of these four numbers are required in row 2 and the sum there must be 45, we deduce that also the two large numbers in cells E1 and E4 have the sum 45. This implies that E3 must be 10.

	A	B	C	D	E	
1	24					60
2	19			11	13	88
3		21			10	40
4			14	6		70
5		12			17	67
	64	84	56	36	85	

I then made a list of all possible solutions for column B. From these I could remove all those where number E3=10 appears and those where two numbers of 20, 22, 23, 25 are needed – these numbers are reserved for cells B2, C2, E1, and E4. I made a corresponding list for row 4.

After these considerations, only two possible solutions for column B remained: 15,16,20 or 8,18,25. From this we can conclude that numbers 22 and 23 (not appearing in these alternatives) must be in column E and one of them must then be in row 4 in cell E4. On the other hand B4 must be less than 20. The only solution for row 4 that appears on my list and fulfills the earlier conditions (either 22 or 23 must be included, both other numbers must be less than 20, number 10 cannot appear in that row) is 9,18,23. From this follows that E3=23 and correspondingly E1=22, and furthermore B4 must be either 9 or 18. Thus the solution for column B is 8,18,25 – B2=25, B4=18 and hence B1=8. C2 is 20.

	A	B	C	D	E	
1	24	8			22	60
2	19	25	20	11	13	88
3		21			10	40
4		18	14	6	23	70
5		12			17	67
	64	84	56	36	85	

Correspondingly A4=9 and A3+A5 must be 12 – from remaining numbers the only alternative is 5+7. Number 7 is too big for A3. This means that A3=5 and A5=7.

	A	B	C	D	E	
1	24	8			22	60
2	19	25	20	11	13	88
3	5	21			10	40
4	9	18	14	6	23	70
5	7	12			17	67
	64	84	56	36	85	

So C5+D5 is 31 with the only permitted alternative 15+16. At this stage it remains open in which order numbers 15 and 16 appear in row 5 and how the small numbers 1, 2, 3, 4 should be located in cells C1, D1, C3, D3.

The final solution is then found easily:

	A	B	C	D	E	
1	24	8	4	2	22	60
2	19	25	20	11	13	88
3	5	21	3	1	10	40
4	9	18	14	6	23	70
5	7	12	15	16	17	67
	64	84	56	36	85	

Also *Anna-Riitta Niskanen* had solved this puzzle.

5. Creating Survo puzzles

It is rather easy to generate Survo puzzles. I have tried to guarantee the uniqueness of the solution by making a general solver program as a Survo module `SUMMAT`.

For example, the previous 5×5 –puzzle was created by means of the Survo matrix interpreter as follows:

```
____________________________________________________________________
FILE CREATE SUMS,10,2
FIELDS:
1 N 2 X         integers 1,2,3,...
2 N 8 Y         random numbers for shuffling the X-numbers
END

FILE INIT SUMS,25
VAR X=ORDER TO SUMS
VAR Y=rand(2006) TO SUMS
m=5 n=5

FILE SORT SUMS BY Y TO SUMS2 / X's to random order
MAT SAVE DATA SUMS2 TO A     / VARS=X  X's as a vector of m*n elements
MAT A=VEC(A,m)               / vector transformed to m*n matrix
MAT A2=ZER(m+1,n+1)          / for collecting elements and marginal sums
MAT A2(1,1)=A
MAT S=SUM(A)
MAT A2(m+1,1)=S              / Sums of columns
MAT S=SUM(A')'
MAT A2(1,n+1)=S              / Sums of rows
MAT S=SUM(S)
MAT A2(m+1,n+1)=S            / m*n*(m*n+1)/2
MAT A=A2
MAT A(1,1)=ZER(m,n)          / Removing elements from A, sums remain
MAT NAME A AS A
```

```
Selecting elements from solution A2:
MAT A(1,1)=A2(1,1)
MAT A(2,1)=A2(2,1)      Some elements were chosen arbitrarily,
MAT A(2,4)=A2(2,4)      the rest of them so that the solver
MAT A(2,5)=A2(2,5)      SUMMAT gives a unique solution.
MAT A(3,2)=A2(3,2)
MAT A(4,3)=A2(4,3)
MAT A(4,4)=A2(4,4)
MAT A(5,2)=A2(5,2)
MAT A(5,5)=A2(5,5)
MAT NAME A AS A

The problem setup is ready:
LOADM A,(2),CUR+1
A
           1   2   3   4   5 Sum
  1       24   0   0   0   0  60
  2       19   0   0  11  13  88
  3        0  21   0   0   0  40
  4        0   0  14   6   0  70
  5        0  12   0   0  17  67
Sum       64  84  56  36  85 325
------------------------------------------------------------------
```

The solver program `SUMMAT` works according to a very elementary, partially randomized algorithm. It starts by inserting the missing numbers randomly in the table and tries then to get the computed sums of rows and columns as close to the true ones as possible by exchanging elements in the table systematically.

This trial leads either to a correct solution or (as in most cases) to dead end where the discrepancy between computed and true sums cannot be diminished systematically. In the latter case a 'mutation' is made by exhanging two or more numbers randomly. Thereafter the systematic procedure plus mutation is repeated until a true solution is found.

In most cases the mean number of mutations works as a crude measure for the level of difficulty of solving a Survo puzzle. This measure (MD) is computed as the mean number of mutations when the puzzle is solved 1000 times by starting from a randomized table. The MD values for the examples 1–4 are 30,20,30, and 560.

The distribution of the number of mutations (as expected) comes close to a geometric distribution. In some cases I have studied which is the optimal number of exchanges in a single mutation. For example, in 5×5 –puzzles three exchanges seems to minimize the number of iterations and the solution is about three times faster than a procedure where each trial is started from scratch.

The solver program includes an option for studying uniqueness of the solution by solving the problem repeatedly and by giving a warning when more than one solution is found.

6. Open Survo puzzles

Puzzles where only row and column sums are given but still have a unique solution are of particular interest. Such puzzles are called *open.* In any case, for example, any open $m \times n$ –puzzle having numbers $1, 2, \ldots mn$ in original order column- or rowwise in its solution has no other solutions. There are also other open puzzles and typically more for larger tables.

Two open Survo puzzles A and B are defined *essentially different* if the solution of A cannot be transformed into the solution of B by interchanging rows and columns or by transposing (in case $m = n$). Let $S(m, n)$ be the number of essentially different open $m \times n$ –Survo puzzles. It is evident that $S(1, 1) = 1$ and $S(2, 2) = 1$.

The only such 2×2 –puzzle is

		3
		7
4	6	

1	2	3
3	4	7
4	6	

since, for example, the puzzle

		3
		7
5	5	

1	2	3
4	3	7
5	5	

2	1	3
3	4	7
5	5	

has two solutions. Thus sums of rows (as well as sums of columns) must be mutually different; otherwise the solution is not unique.

Reijo Sund was the first to study these open Survo puzzles and he found that the number of essentially different open 3×3 –puzzles is 38 and thus $S(3, 3) = 38$. By starting from all possible 9!=362880 cases he achieved this result by means of clever combinations of typical Survo basic operations (Appendix 1).

The table below gives the row and column sums and degrees of difficulty (MD) of these puzzles.

Rows			Columns			MD	Rows			Columns			MD
6	17	22	11	14	20	16	9	16	20	8	14	23	15
7	14	24	11	15	19	13	9	16	20	8	15	22	23
7	15	23	10	15	20	21	10	13	22	8	16	21	22
7	16	22	9	16	20	28	10	14	21	7	16	22	15
7	16	22	10	15	20	26	10	14	21	8	14	23	25
7	17	21	9	15	21	16	10	15	20	6	18	21	24
7	17	21	10	15	20	26	10	15	20	9	12	24	18
7	17	21	11	14	20	46	10	15	20	9	13	23	31
7	17	21	11	16	18	47	10	16	19	6	17	22	22
8	14	23	10	15	20	21	10	16	19	7	15	23	13
8	15	22	8	17	20	15	10	16	19	8	13	24	16
8	15	22	9	17	19	30	11	14	20	7	15	23	19
8	15	22	10	13	22	15	11	14	20	8	13	24	20
8	15	22	10	14	21	35	11	15	19	6	16	23	18
8	15	22	11	13	21	47	11	15	19	7	15	23	32
9	13	23	9	15	21	20	11	16	18	7	15	23	31
9	13	23	10	16	19	36	12	14	19	7	15	23	29
9	14	22	8	17	20	23	12	14	19	9	13	23	59
9	15	21	8	15	22	31	12	15	18	6	15	24	19

Although MD values given by the solver program should be considered with precaution, in any case the last but one with MD=59 is one of the most difficult. It is solved by the following steps (quotation from a Survo edit field):

```
____________________________________________________________________
The puzzle to be solved is
  A  B  C
1 *  *  * 12
2 *  *  * 14
3 *  *  * 19
  9 13 23

Let's start from column C (the only sum with unique partition):
COMB P,CUR+1 / P=PARTITIONS,23,3 DISTINCT=1 MAX=9
Partitions 3 of 23: N[P]=1
6 8 9

  A  B  C
1 *  *  6 12    C column numbers not necessarily in correct order
2 *  *  8 14
3 *  *  9 19
  9 13 23

......................................................................
Column B:
OFF=6,8,9
COMB P,CUR+1 / P=PARTITIONS,13,3 DISTINCT=1 MAX=9
Partitions 3 of 13: N[P]=2
1 5 7
2 4 7

Number 7 in column B and in row 2 or 3
- not in row 1 since sum 12 would be exceeded

......................................................................
Column A:
OFF=8,6,9,7
COMB P,CUR+1 / P=PARTITIONS,9,3 DISTINCT=1 MAX=9
Partitions 3 of 9: N[P]=2
1 3 5
2 3 4

Number 3 is in column A.
......................................................................
```

```
Is B2=7?
Then it would be C2=6 and A2=1

   A  B  C
1  *  *  8 12
2  1  7  6 14
3  *  *  9 19
   9 13 23

and in column A 9=3+1+5, but the row sum 12 would be exceeded!
Thus B3=7.

   A  B  C
1  *  *  6 12
2  *  *  8 14
3  *  7  9 19
   9 13 23

The only partition of two first numbers of of column B is
13-7=6=1+5 - not 2+4 since this leads to an immediate contradiction
in the first row.
Then we get B1=1 and B2=5
since either B1=5 exceeds row sum 12 or, if C1=6, A1 would be 1.

   A  B  C
1  *  1  6 12
2  *  5  8 14
3  *  7  9 19
   9 13 23

.......................................................................
Since B2=5, C2 cannot be 8 or 9 - Thus C2=6 and A2=3.

   A  B  C
1  *  1  8 12
2  3  5  6 14
3  *  7  9 19
   9 13 23

Then inevitably A1=2 and A3=4 and then C1=9 and C3=8

   A  B  C
1  2  1  9 12
2  3  5  6 14
3  4  7  8 19
   9 13 23
--------------------------------------------------------------------
```

Enumeration of essentially different open Survo puzzles becomes more difficult for larger m and n combinations. By using my solver program I have got S(3,4)=583. Because the number of possible puzzles in then 12!=479001600, it is more profitable to study the problem through all possible combinations of marginal sums and then the number of possible candidates is reduced to $128 \times 519 = 66432$ revealed by the following calculation:

```
-------------------------------------------------------------------
The sum of all numbers in a 3x4-table is 12*13/2=78.
COMB P,CUR+1 / P=PARTITIONS,78,3 DISTINCT=1 MIN=10 MAX=42 RESULTS=0
Partitions 3 of 78: N[P]=128

COMB P,CUR+1 / P=PARTITIONS,78,4 DISTINCT=1 MIN=6  MAX=33 RESULTS=0
Partitions 4 of 78: N[P]=519
-------------------------------------------------------------------
```

Enumeration of essentially different open 4×4 –puzzles requires more advanced methods since the number of possible marginal sum combinations is $2980 \times (2980 - 1)/2 = 4438710$ due to

```
-------------------------------------------------------------------
16*17/2=136                   1+2+3+4=10     58=13+14+15+16
COMB P,CUR+1 / P=PARTITIONS,136,4 MIN=10 MAX=58 DISTINCT=1 RESULTS=0
Partitions 4 of 136: N[P]=2980
-------------------------------------------------------------------
```

Petteri Kaski obtained $S(4,4) = 5327$ using a backtrack algorithm for the exact cover problem together with additional constraints for the marginal sums.

Although solving an open Survo puzzle is usually a demanding task, sometimes the solution is found by a smart initial observation. *Kimmo Vehkalahti* solved

	A	B	C	D	E	F	
1							38
2							40
	8	9	10	16	17	18	

(MD=780) as follows: Let's study column sets A,B,C and D,E,F pairwise and their simultaneous possibilities:

	A	B	C		D	E	F
1	1+7	3+6	2+8	**a**	**4+12**	**6+11**	**8+10**
2	1+7	4+5	2+8	b	5+11	7+10	6+12
3	2+6	1+8	3+7	c	5+11	8+9	6+12
4	2+6	4+5	1+9	d	6+10	5+12	7+11
5	3+5	1+8	4+6	e	6+10	8+9	7+11
6	**3+5**	**2+7**	**1+9**	f	7+9	5+12	8+10
7	3+5	2+7	4+6				

It is immediately seen that the only proper combination is 6a, i.e.

A	B	C		D	E	F
3+5	2+7	1+9	and	4+12	6+11	8+10

By listing all partitions of the first row (55 cases)

```
____________________________________________________________________
COMB P,CUR+1 / P=PARTITIONS,38,6 DISTINCT=1 MAX=12
Partitions 6 of 38: N[P]=55
1 2 3 9 11 12
1 2 4 8 11 12
. . .
3 4 6 7 8 10
3 5 6 7 8 9
____________________________________________________________________
```

and by removing all such partitions where numbers 3,5 or 2,7 or 1,9 or 4,12 or 6,11 or 8,10 appear simultaneously only one possibility remains giving the solution

	A	B	C	D	E	F	
1	5	7	1	4	11	10	38
2	3	2	9	12	6	8	40
	8	9	10	16	17	18	

7. Solution of one 4×4 –puzzle

The only Survo puzzle of those presented in the Survo users newsgroup (in April and May 2006) getting no solution was

	A	B	C	D	
1					14
2					28
3					44
4					50
	16	33	40	47	

The solution is again described in terms of Survo.

```
____________________________________________________________________
We are solving

    A   B   C   D
1   *   *   *   *  14
2   *   *   *   *  28
3   *   *   *   *  44
4   *   *   *   *  50
5  16  33  40  47

having MD=1150.

At first it is shown that A1=1.
The first row is studied:
Partitions 14,4:
COMB P,CUR+1 / P=PARTITIONS,14,4 DISTINCT=1 MAX=16
Partitions 4 of 14: N[P]=5
1 2 3 8
1 2 4 7
1 2 5 6
1 3 4 6
2 3 4 5
On this basis A1 is one of numbers 1,2,3,4,5,6,7,8.
By considering possible partitions of column A (sum 16)
alternatives 3,4,5,6,7,8 are cancelled.
The case A1=3 as an example:
.......................................................................
Let's go through all partitions 16,4 of row 1:
COMB P,CUR+1 / P=PARTITIONS,16,4 DISTINCT=1  OFF=1,2,8
Partitions 4 of 16: N[P]=0
COMB P,CUR+1 / P=PARTITIONS,16,4 DISTINCT=1  OFF=1,4,6
Partitions 4 of 16: N[P]=0
COMB P,CUR+1 / P=PARTITIONS,16,4 DISTINCT=1  OFF=2,4,5
Partitions 4 of 16: N[P]=0
Thus A1 cannot be 3.
```

```
.......................................................................
Case A1=2:
According to partition 14,4 above the three last numbers in row 1 would be
1,3,8 or 1,4,7 or 1,5,6 or 3,4,5 (each in some order).

COMB P,CUR+1 / P=PARTITIONS,16,4 DISTINCT=1  OFF=1,3,8
Partitions 4 of 16: N[P]=0
COMB P,CUR+1 / P=PARTITIONS,16,4 DISTINCT=1  OFF=1,4,7
Partitions 4 of 16: N[P]=1
2 3 5 6

    A   B   C   D
1   2   1   4   7  14      The table could be like this.
2   3   *   *   *  28      Red numbers in row 1 and in
3   5   *   *   *  44      column A are not necessarily
4   6   *   *   *  50      in the right order.
5  16  33  40  47

If now the smallest of unused numbers 8,9,10 are tried
to put in row 2

    A   B   C   D
1   2   1   4   7  14
2   3   8   9  10  28
3   5   *   *   *  44
4   6   *   *   *  50
5  16  33  40  47

the row sum becomes 3+8+9+10=30>28 and is too high.

COMB P,CUR+1 / P=PARTITIONS,16,4 DISTINCT=1 MAX=16  OFF=1,5,6
Partitions 4 of 16: N[P]=1
2 3 4 7

This also leads exceedence of the sum 28.

COMB P,CUR+1 / P=PARTITIONS,16,4 DISTINCT=1 MAX=16  OFF=3,4,5
Partitions 4 of 16: N[P]=1
1 2 6 7

    A   B   C   D
1   2   3   4   5  14
2   1   8   9  10  28
3   6   *   *   *  44
4   7   *   *   *  50
5  16  33  40  47

Now the second row sum would be correct, but although
the greatest possible numbers 5 and 10 are put in column D,
the difference 47-5-10=32 is too much to be presented
as a sum of two permitted integers.

Thus A1=1.
```

```
.......................................................................
    A   B   C   D
1   1   *   *   *  14
2   *   *   *   *  28
3   *   *   *   *  44
4   *   *   *   *  50
5  16  33  40  47

Partitions of row 1 and column A are studied further - now with restriction A1=1.
14-1=13
COMB P,CUR+1 / P=PARTITIONS,13,3 DISTINCT=1 MAX=16  OFF=1
Partitions 3 of 13: N[P]=4
2 3 8
2 4 7
2 5 6
3 4 6

16-1=15
COMB P,CUR+1 / P=PARTITIONS,15,3 DISTINCT=1 MAX=16  OFF=1
Partitions 3 of 15: N[P]=7
2 3 10
2 4 9
2 5 8
2 6 7
3 4 8
3 5 7
4 5 6

Possible pairs of partitions are then
14-1    16-1
2 3 8   4 5 6
2 5 6   3 4 8
3 4 6   2 5 8

and so numbers 1,2,3,4,5,6,8 are in row 1 and column A.
The smallest free numbers to columns B,C,D of row 2 are
then 7,9,10. Since their sum 7+9+10=26 remains 2 units
below the sum 28, we have A2=2.
All other free numbers would exceed the row sum.

The current situation is
    A   B   C   D
1   1   *   *   *  14    3,4,6  in this row in some order
2   2   *   *   *  28    7,9,10 in this row in some order
3   *   *   *   *  44
4   *   *   *   *  50
5  16  33  40  47

5,8 in column A in some order.
.......................................................................
The possible partitions of column B are listed:
COMB P,CUR+1 / P=PARTITIONS,33,4 DISTINCT=1 MAX=16  OFF=1,2,5,8
Partitions 4 of 33: N[P]=17
```

```
3 4 10 16
3 4 11 15
3 4 12 14
3 6 9 15
3 6 10 14
3 6 11 13
3 7 9 14
3 7 10 13
3 7 11 12
3 9 10 11
4 6 7 16
4 6 9 14
4 6 10 13
4 6 11 12
4 7 9 13
4 7 10 12
6 7 9 11
```

```
Only allowed partitions are those containing not simultaneously
numbers 3,4,6 or on the other hand numbers 7,9,10.
The only possibility is then 3+7+11+12=33 and according to
already found partitions of the two first rows we have
B1=3 and B2=7.

We have come to a situation where numbers indicated by the same
color are not necessarily in the correct order:

    A   B   C   D
1   1   3   4   6  14
2   2   7   9  10  28
3   5  11  13  14  44
4   8  12  15  16  50
5  16  33  40  47

To find the correct order, possible partitions of column D
are listed:
....................................................................
COMB P,CUR+1 / P=PARTITIONS,47,4 DISTINCT=1 MAX=16 OFF=1,2,3,7,5,8,11,12
Partitions 4 of 47: N[P]=3
4 13 14 16
6 10 15 16
9 10 13 15

Only the second one 6+10+15+16=47 is possible according to what is known
about the two first rows and the ambiguities of the table are reduced
into the form

    A   B   C   D
1   1   3   4   6  14
2   2   7   9  10  28
3   5  11  13  15  44
4   8  12  14  16  50
5  16  33  40  47
```

```
and this happens to be the correct solution.
In principle, for the two last rows there are still
2^4=16 alternatives which is considerably less than the original
number fact(12)=479001600.

The uniqueness of the solution is confirmed most easily by looking for
the possible partitions of the last row:
COMB P,CUR+1 / P=PARTITIONS,50,4 DISTINCT=1 MAX=16 OFF=1,3,4,6,2,7,9,10
Partitions 4 of 50: N[P]=5
5 14 15 16
8 11 15 16
8 12 14 16
8 13 14 15
11 12 13 14

From these alternatives only 8+12+14+16=50 is composed
in a valid manner.
--------------------------------------------------------------------
```

8. Problems

Twelve Survo puzzles are presented. Solutions of 10 first of them are also given. The number in parentheses tells the degree of difficulty (MD) according to the solver program. These numbers should be considered with a precaution since they do not take into account creative insights typical for human reasoning.

Problem 1 (1.5)

			7
			14
9	8	4	

Problem 2 (18)

				11
				25
8	11	7	10	

Problem 3 (17)

12				28
	8		1	28
		6		22
32	16	23	7	

Problem 4 (50)

			8
			15
			22
11	13	21	

Problem 5 (50)

10				12	2		63
	17		11		3	6	89
19		5					79
49	41	32	30	27	18	34	

Problem 6 (105)

	23		1		18	93
2		3		16		67
	10			13		67
15		11	14		8	73
46	61	39	42	45	67	

Problem 7 (30)

12		8		11	50
	13		10		47
		2			23
24	23	13	34	26	

Problem 8 (15)

13			6	26
	9			43
		12		26
1			10	41
30	28	47	31	

Problem 9 (55)

12				19			10			106
					17			14	8	104
16	22	6	25	39	20	24	12	23	23	

Problem 10 (145)

				24
				15
				39
21	10	18	29	

Problem 11 (2050)

				51
				36
				32
				17
51	42	26	17	

Problem 12 (17000 "Beast")

10		29			5	83
	27			8		94
33		19	32		22	139
		21	12			101
	20			28		86
30		11			35	163
93	141	87	99	112	134	666

9. Solutions

Problem 1 (1.5)

4	2	1	7
5	6	3	14
9	8	4	

Problem 2 (18)

1	5	3	2	11
7	6	4	8	25
8	11	7	10	

Problem 3 (17)

12	5	7	4	28
9	8	10	1	28
11	3	6	2	22
32	16	23	7	

Problem 4 (50)

3	1	4	8
2	5	8	15
6	7	9	22
11	13	21	

Problem 5 (50)

10	8	9	15	12	2	7	63
20	17	18	11	14	3	6	89
19	16	5	4	1	13	21	79
49	41	32	30	27	18	34	

Problem 6 (105)

20	23	19	1	12	18	93
2	7	3	22	16	17	67
9	10	6	5	13	24	67
15	21	11	14	4	8	73
46	61	39	42	45	67	

Problem 7 (30)

12	4	8	15	11	50
7	13	3	10	14	47
5	6	2	9	1	23
24	23	13	34	26	

Problem 8 (15)

13	3	4	6	26
11	9	15	8	43
5	2	12	7	26
1	14	16	10	41
30	28	47	31	

Problem 9 (55)

12	6	1	18	19	3	13	10	9	15	106
4	16	5	7	20	17	11	2	14	8	104
16	22	6	25	39	20	24	12	23	23	

Problem 10 (145)

7	3	5	9	24
4	1	2	8	15
10	6	11	12	39
21	10	18	29	

APPENDIX 1: ANALYSIS OF 3×3 –PUZZLES

REIJO SUND
19 APRIL 2006

In the following extract from a Survo edit field it is shown what are the essentially different open 3×3 – puzzles.

```
--------------------------------------------------------------------
All possible 3x3-tables are generated (N=362880):

FILE DEL SRIS.TXT
COMB P TO SRIS.TXT / P=PERMUTATIONS,9 RESULTS=1
FILE DEL SRIS01
FILE SAVE SRIS.TXT TO NEW SRIS01 / NEWSPACE=80,20

.......................................................................
The following symbolic notation is used for a table:

X1 X2 X3 R1
X4 X5 X6 R2
X7 X8 X9 R3
S1 S2 S3

.......................................................................
Row and column sums are computed and each configuration of them
is also coded into a single characteristic variable JARJ:

VAR R1:1,R2:1,R3:1,S1:1,S2:1,S3:1,JARJ:8 TO SRIS01
R1=X1+X2+X3
R2=X4+X5+X6
R3=X7+X8+X9
S1=X1+X4+X7
S2=X2+X5+X8
S3=X3+X6+X9
JARJ=S3+100*S2+10000*S1+1000000*R3+100000000*R2+JARJ2
JARJ2=10000000000*R1+1000000000000

FILE SHOW SRIS01
.......................................................................
Data is sorted according to JARJ:

FILE SORT SRIS01 BY JARJ TO SRIS02
.......................................................................
```

```
The number of various row and column sum combinations
is computed (N=46147):

FILE AGGR SRIS02 BY JARJ TO SRIS03 / PRIND=0
VARIABLES:
JARJ FIRST JARJ
MAARA N JARJ
X1 FIRST X1
X2 FIRST X2
X3 FIRST X3
X4 FIRST X4
X5 FIRST X5
X6 FIRST X6
X7 FIRST X7
X8 FIRST X8
X9 FIRST X9
R1 FIRST R1
R2 FIRST R2
R3 FIRST R3
S1 FIRST S1
S2 FIRST S2
S3 FIRST S3
END

FILE SHOW SRIS03
......................................................................
Tables having a unique solution (N=2736) are copied to a new file:

FILE COPY SRIS03 TO NEW SRIS04 / IND=MAARA,1 NEWSPACE=120,30

FILE SHOW SRIS04
......................................................................
Now the essentially different tables should be found by
selecting one representative for each type at random.

Sorting by row and column sums:

VARSTAT SRIS04,*,SORT / VARS=R1,R2,R3
VARSTAT SRIS04,*,SORT / VARS=S1,S2,S3

......................................................................
Characteristic variables for row and column sum combinations
as well as a random variable are generated:

VAR RJARJ:8,SJARJ:8,SAT TO SRIS04
RJARJ=R3+100*R2+10000*R1
SJARJ=S3+100*S2+10000*S1
SAT=rnd(999)

......................................................................
```

```
Rows and columns may be exchanged:

VARSTAT SRIS04,*,SORT / VARS=RJARJ,SJARJ

......................................................................
Sorting by characteristic variables and within similar cases randomly:

VAR JARJ:8=SJARJ+1000000*RJARJ+1000000000000 TO SRIS04
FILE SORT SRIS04 BY JARJ,SAT TO SRIS05

......................................................................
A representative of each essentially different table is selected (N=38):

FILE AGGR SRIS05 BY JARJ TO SRIS06 / PRIND=0
VARIABLES:
JARJ FIRST JARJ
MAARA N JARJ
X1 FIRST X1
X2 FIRST X2
X3 FIRST X3
X4 FIRST X4
X5 FIRST X5
X6 FIRST X6
X7 FIRST X7
X8 FIRST X8
X9 FIRST X9
END

FILE SHOW SRIS06
......................................................................
Row and column sums are computed again:

FILE EXPAND SRIS06,30,120
VAR R1:1,R2:1,R3:1,S1:1,S2:1,S3:1 TO SRIS06
R1=X1+X2+X3
R2=X4+X5+X6
R3=X7+X8+X9
S1=X1+X4+X7
S2=X2+X5+X8
S3=X3+X6+X9
```

```
All essentially different tables are displayed (N=38):

FILE SORT SRIS06 BY R1,R2,R3,S1,S2,S3 TO SRIS06S
FILE LOAD SRIS06S
```

X1	X2	X3	X4	X5	X6	X7	X8	X9	R1	R2	R3	S1	S2	S3
7	9	6	5	8	4	2	3	1	6	17	22	11	14	20
3	6	5	7	9	8	1	4	2	7	14	24	11	15	19
8	6	9	2	1	4	5	3	7	7	15	23	10	15	20
3	6	7	5	8	9	1	2	4	7	16	22	9	16	20
9	5	8	4	2	1	7	3	6	7	16	22	10	15	20
7	5	9	6	3	8	2	1	4	7	17	21	9	15	21
4	1	2	7	6	8	9	3	5	7	17	21	10	15	20
9	5	3	7	8	6	4	1	2	7	17	21	11	14	20
3	5	9	6	7	8	2	4	1	7	17	21	11	16	18
1	5	2	6	8	9	3	7	4	8	14	23	10	15	20
3	4	1	6	7	2	8	9	5	8	15	22	8	17	20
8	2	5	4	1	3	7	6	9	8	15	22	9	17	19
7	6	9	4	3	8	2	1	5	8	15	22	10	13	22
3	4	1	5	8	2	6	9	7	8	15	22	10	14	21
9	6	7	8	2	5	4	3	1	8	15	22	11	13	21
4	2	7	3	1	5	8	6	9	9	13	23	9	15	21
9	8	6	5	7	1	2	4	3	9	13	23	10	16	19
8	5	9	6	1	7	3	2	4	9	14	22	8	17	20
4	9	8	1	6	2	3	7	5	9	15	21	8	15	22
5	8	3	7	9	4	2	6	1	9	16	20	8	14	23
9	3	8	7	4	5	6	1	2	9	16	20	8	15	22
9	8	5	3	6	1	4	7	2	10	13	22	8	16	21
2	5	7	4	8	9	1	3	6	10	14	21	7	16	22
3	1	6	4	2	8	7	5	9	10	14	21	8	14	23
5	4	1	7	6	2	9	8	3	10	15	20	6	18	21
2	7	1	4	8	3	6	9	5	10	15	20	9	12	24
5	2	8	1	3	6	7	4	9	10	15	20	9	13	23
9	7	3	5	4	1	8	6	2	10	16	19	6	17	22
8	7	4	6	3	1	9	5	2	10	16	19	7	15	23
5	8	3	6	9	4	2	7	1	10	16	19	8	13	24
5	8	1	7	9	4	3	6	2	11	14	20	7	15	23
6	5	9	4	2	8	3	1	7	11	14	20	8	13	24
9	3	7	6	1	4	8	2	5	11	15	19	6	16	23
4	8	7	2	6	3	1	9	5	11	15	19	7	15	23
5	4	9	7	1	8	3	2	6	11	16	18	7	15	23
3	2	9	7	4	8	5	1	6	12	14	19	7	15	23
1	2	9	7	4	8	5	3	6	12	14	19	9	13	23
3	9	6	2	8	5	1	7	4	12	15	18	6	15	24

APPENDIX 2: ABOUT RESTRICTED INTEGER PARTITIONS

SEPPO MUSTONEN
20 AUGUST 2006

Integer partitions play a prominent role when solving Survo puzzles. In this appendix it is shown how Survo can be used for studying and enumerating restricted integer partitions, especially those with distinct parts. Most of the results are well-known from the standard literature on the subject.

1. Basic definitions and relations

Let $\mathbf{P}(n, m)$ be the set of partitions of integer n into m parts and $P(n, m)$ be the number of these partitions. For example, the following `COMB` command lists elements of $\mathbf{P}(10, 3)$ in lexicographic order and gives $P(10, 3) = 8$.

```
____________________________________________________________________
COMB P,CUR+1 / P=PARTITIONS,10,3
Partitions 3 of 10: N[P]=8
1 1 8
1 2 7
1 3 6
1 4 5
2 2 6
2 3 5
2 4 4
3 3 4
____________________________________________________________________
```

Let $\mathbf{Q}(n, m)$ be the set of partitions of integer n into m distinct parts and $Q(n, m)$ be the number of these partitions. Using `COMB` with a specification `DISTINCT=1` we get the elements of $\mathbf{Q}(13, 3)$ as

```
____________________________________________________________________
COMB Q,CUR+1 / Q=PARTITIONS,13,3 DISTINCT=1
Partitions 3 of 13: N[P]=8
1 2 10
1 3 9
1 4 8
1 5 7
2 3 8
2 4 7
2 5 6
3 4 6
____________________________________________________________________
```

Thus $Q(13, 3) = P(10, 3) = 8$ and there is also a simple relation between the elements of $\mathbf{Q}(13, 3)$ and $\mathbf{P}(10, 3)$ since any element of $\mathbf{P}(10, 3)$, say (p_1, p_2, p_3), corresponds to $(p_1, p_2 + 1, p_3 + 2)$ in $\mathbf{Q}(13, 3)$.

In general, we have

$$P(n,m) = Q(n + m(m-1)/2, m) \tag{1}$$

and this is proved easily (by following the idea given in the previous example) by establishing one-to-one correspondence between the elements $(p_1, p_2, p_3, \ldots, p_m)$ of $\mathbf{P}(n,m)$ and $(p_1, p_2 + 1, p_3 + 2, \ldots, p_m + m - 1)$ of $\mathbf{Q}(n + m(m-1)/2, m)$.

Let $\mathbf{R}(n,m)$ be the set of partitions of integer n into at most m parts and $R(n,m)$ be the number of these partitions. The elements of the set $\mathbf{R}(7,3)$ are listed by

```
____________________________________________________________________
COMB R,CUR+1 / R=PARTITIONS,7,1
Partitions 1 of 7: N[R]=1
7
COMB R,CUR+1 / R=PARTITIONS,7,2
Partitions 2 of 7: N[R]=3
1 6
2 5
3 4
COMB R,CUR+1 / R=PARTITIONS,7,3
Partitions 3 of 7: N[R]=4
1 1 5
1 2 4
1 3 3
2 2 3
____________________________________________________________________
```

Thus we have $R(7,3) = 1 + 3 + 4 = 8 = P(10,3)$ and in general

$$P(n,m) = R(n-m, m). \tag{2}$$

Also this can be proved by observing that there is a one-to-one mapping from the elements $(p_1, p_2, \ldots, p_n)$ of $\mathbf{P}(n,m)$ to the elements $(p_1 - 1, p_2 - 1, \ldots, p_n - 1)$ of $\mathbf{R}(n-m, m)$.

Let $\mathbf{S}(n,m)$ be the set of partitions of integer n into parts with m as the greatest part and $S(n,m)$ be the number of these partitions. The elements of the set $\mathbf{S}(10,3)$ are obtained in Survo by

```
____________________________________________________________________
COMB S,CUR+1 / S=PARTITIONS,10 GREATEST=3
Partitions of 10: N[S]=8
1 3 3 3
2 2 3 3
1 1 2 3 3
1 2 2 2 3
1 1 1 1 3 3
1 1 1 2 2 3
1 1 1 1 1 2 3
1 1 1 1 1 1 1 3
____________________________________________________________________
```

We have $S(10,3) = 8 = P(10,3)$ and in general

$$P(n,m) = S(n,m). \tag{3}$$

since there is again a bijective mapping from $\mathbf{S}(n,m)$ to $\mathbf{P}(n,m)$ shown by means of Ferrers graphs (see i.e. [2]). For example, the Ferrers graph of the third partition above (1,1,2,3,3) is presented as

1 •
1 •
2 • •
3 • • •
3 • • •

By transposing the graph we get
2 • •
3 • • •
5 • • • • •

which represents the third last element (2,3,5) in the set $\mathbf{P}(10,3)$.

Let $\mathbf{T}(n,m)$ be the set of partitions of integer n into parts with parts $\leq m$ and $T(n,m)$ be the number of these partitions. The elements of the set $\mathbf{T}(10,3)$ are obtained in Survo by

```
____________________________________________________________________
COMB T,CUR+1 / T=PARTITIONS,7 MAX=3
Partitions of 7: N[T]=8
1 3 3
2 2 3
1 1 2 3
1 2 2 2
1 1 1 1 3
1 1 1 2 2
1 1 1 1 1 2
1 1 1 1 1 1 1
____________________________________________________________________
```

We have $T(7,3) = 8 = S(10,3)$ and in general

$$T(n-m,m) = S(n,m) = P(n,m). \tag{4}$$

This is true since there is a bijection between $\mathbf{T}(n-m,m)$ and $\mathbf{S}(n,m)$ which is seen by removing the largest part m from each element of $\mathbf{S}(n,m)$. Then by (1) and (4) we have

$$Q(n,m) = T(n-m(m+1)/2,m). \tag{5}$$

The $Q(n,m)$ numbers have a simple recursion formula

$$Q(n,m) = Q(n-m,m) + Q(n-m,m-1) \tag{6}$$

and its proof is based on the following observation. Let $n = q_1 + q_2 + \cdots + q_m$ be a partition of n into distinct parts $q_1 < q_2 < \cdots < q_m$. Then $\mathbf{Q}(n,m)$ is a union of two disjoint subsets $\mathbf{Q}_1(n,m)$ and $\mathbf{Q}_2(n,m)$ where $\mathbf{Q}_1(n,m)$ includes partitions with $q_1 > 1$. Since $n-m = (q_1-1)+(q_2-1)+\cdots+(q_m-1)$ and $q_2 > 1$, we have

$$Q(n,m) = |\mathbf{Q}_1(n,m)| + |\mathbf{Q}_2(n,m)| = Q(n-m,m) + Q(n-m,m-1).$$

2. Linear recurrence equation

The numbers $T(n,m)$ (and similarly numbers $P(n,m)$, $Q(n,m)$, $R(n,m)$, and $S(n,m)$) can be presented in the form

$$T(n,m) = 1 + a_1T(n-1,m) + a_2T(n-2,m) + \cdots + a_kT(n-k,m) \tag{7}$$

where $k = m(m+1)/2 - 1$. To show this let's study the generating function $g(s)$ of numbers $f(n)$ defined by

$$f(n) = 1 + a_1f(n-1) + a_2f(n-2) + \cdots + a_kf(n-k), \quad n = 0, 1, 2, \ldots.$$

Then we have

$$g(s) = \sum_{n=0}^{\infty} f(n)s^n = 1/(1-s) + a_1sg(s) + a_2s^2g(s) + \cdots + a_ks^kg(s)$$

giving

$$g(s) = \frac{1}{(1-s)(1-a_1s-a_2s^2-\cdots-a_ks^k)}. \tag{8}$$

On the other hand, the generating function $g_m(s)$ of the $T(n,m)$ numbers is simply

$$\begin{aligned} g_m(s) &= (1+s+s^2+\ldots)(1+s^2+s^4+\ldots)\cdots(1+s^m+s^{2m}+\ldots) \\ &= \frac{1}{(1-s)(1-s^2)\cdots(1-s^m)}, \end{aligned} \tag{9}$$

which is of the form (8) and then the coefficients $a_1, a_2, \ldots a_k$ are obtained by expanding the product $(1-s^2)(1-s^3)\cdots(1-s^m)$ to a polynomial.

By (5) the generating function of the $Q(n,m)$ numbers is $s^{m(m+1)/2}g_m(s)$ and so the same coefficients $a_1, a_2, \ldots a_k$ are valid for the recurrence formula of $Q(n,m)$.

In the following, the coefficients $a_1, a_2, \ldots$ are evaluated for $m = 5$ in Survo by `MAT` and `POL` commands.

```
____________________________________________________________________
MAT P2=ZER(3,1)
MAT P2(1,1)=1
MAT P2(3,1)=-1   / Polynomial 1-s^2 as a vector (1,0,-1)
MAT P3=ZER(4,1)
MAT P3(1,1)=1
MAT P3(4,1)=-1   / Polynomial 1-s^3 as a vector (1,0,0,-1)
MAT P4=ZER(5,1)
MAT P4(1,1)=1
MAT P4(5,1)=-1   / Polynomial 1-s^4 as a vector (1,0,0,0,-1)
MAT P5=ZER(6,1)
MAT P5(1,1)=1
MAT P5(6,1)=-1   / Polynomial 1-s^5 as a vector (1,0,0,0,0,-1)
POL P=P2*P3
POL P=P*P4
POL P=P*P5       / (1-s^2)(1-s^3)(1-s^4)(1-s^5) as a polynomial

MAT LOAD P',##,CUR+1
MATRIX P'
Polynom'
///       0  1  2  3  4  5  6  7  8  9 10 11 12 13 14
real      1  0 -1 -1 -1  0  1  2  1  0 -1 -1 -1  0  1
____________________________________________________________________
```

The case $m = 6$ follows similarly.

```
---------------------------------------------------------------------
MAT P6=ZER(7,1)
MAT P6(1,1)=1
MAT P6(7,1)=-1   / Polynomial 1-s^6 as a vector (1,0,0,0,0,0,-1)
POL P=P*P6       / (1-s^2)(1-s^3)(1-s^4)(1-s^5)(1-s^6) as a polynomial

MAT LOAD P',##,CUR+1
MATRIX P'
Polynom'
///       0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20
real      1  0 -1 -1 -1  0  0  2  2  1  0 -1 -2 -2  0  0  1  1  1  0 -1
---------------------------------------------------------------------
```

Thus, for example, $f(n) = Q(n, 5)$ (as well as $P(n, 5)$, $R(n, 5)$, $S(n, 5)$, and $T(n, 5)$) has the recurrent representation

$$\begin{aligned} f(n) = &1 + f(n-2) + f(n-3) + f(n-4) - f(n-6) - 2f(n-7) \\ &- f(n-8) + f(n-10) + f(n-11) + f(n-12) - f(n-14). \end{aligned} \tag{10}$$

3. Explicit expressions

No general formulas for $Q(n, m)$ (and other related functions) are known. However, for any fixed value of m, it is possible to compute formulas for them [1], [2]. Those formulas are typically derived by using the generating function (9).

Here an alternative computational approach based on Survo leads to formulas for $Q(n, m)$ when $m \leq 6$. On the assumption that the $Q(n, m)$ can be expressed essentially as a polynomial of degree $m - 1$, the coefficients of the polynomial are estimated from a large data set consisting of successive $Q(n, m)$ values. These values are computed easily by using the recursion formula (7).

As an example, the case $m = 5$ is studied. I will be shown as a Survo application that

$$Q(n, 5) = \lceil (n(n^3 - 30n^2 + 400n - 1320 - 180\lfloor n/2 \rfloor) + 1)/2880 \rceil.$$

All the essential steps of the proof are provided in the following, rather lengthy quotation from a Survo edit field.

```
-----------------------------------------------------------------
*SAVE PART5  / Q(m,5)
*
*Creating a data file PARTS5 for n and Q(n,5) values
*
*FILE CREATE PARTS5,64,8
*FIELDS:
*1 N 8 n          (###########)
*2 N 8 a          (#################) Q(n,5)
*END
*
*FILE INIT PARTS5,70000 / Space for 70000 observations
*
*VAR n,a TO PARTS5 / Setting n values and initializing Q(n,5) as zeros
*n=ORDER a=0
*................................................................
*Computing Q(n,5) values by the recursion formula (10):
*To avoid possible overflows, terms are reordered in the calculation.
*VAR a TO PARTS5 / IND=ORDER,15,70000 ( Q(n,5)=0, for n<15, Q(15,5)=1 )
```

```
*a=a1+a2+a3-a4+a5+a[-12]
*a1=1+a[-2]-a[-6]
*a2=a[-3]-a[-7]
*a3=a[-4]-a[-7]
*a4=a[-8]-a[-10]
*a5=a[-11]-a[-14]
*.....................................................................
*Since preliminary calculations had indicated that the Q(n,5) formula will
*depend on the oddness of n, the parity of n is computed as a variable I.
*VAR I:1=mod(n,2) TO PARTS5
*.....................................................................
*It is assumed that a=Q(n,5) can be presented essentially
*as a polynomial of the 4th degree.
*A model for that polynomial is defined here in a computationally
*feasible form.
*MODEL M5
*a=c0+n*(c1+n*(c2+n*(c3+n*c4)))
*.....................................................................
*The non-zero values are copied to a new file PART5:
*FILE COPY PARTS5 TO NEW PART5 / IND=ORDER,15,70000
*.....................................................................
*The parameters c0,c1,c2,c3,c4 are estimated by the ESTIMATE operation
*for even n values (I=0):
*Initial values of the parameters are zero.
*The first activation of ESTIMATE leads to better initial values.
*The second activation gives the results
*ACCURACY=16  IND=I,0    METHOD=N
*ESTIMATE PART5,M5,CUR+1  / RESULTS=0  PRIND=0  / Activated twice!
*Estimated parameters of model M5:
*c0=0.705973361793440 (0.011434969248571)
*c1=-0.458334232603667 (0.000002261988330)
*c2=0.107638888954769 (0.000000000131339)
*c3=-0.010416666666668 (0.000000000000003)
*c4=0.000347222222222 (0.000000000000000)
*n=34993 rss=6370.837946 R^2=1.00000 nf=22
*
*Studying parameters by calculations
*1/c4=2880.0000000018435
*2880*c3=-30.00000000000384
*2880*c2=310.00000018973475
*2880*c1=-1320.0025898985609
*gives a preliminary suggestion
*Q(n,5)=n*(-1320+n*(310+n*(-30+n)))/2880 + eps  if mod(n,2)=0
*where eps is a residual term to be studied later.
*.....................................................................
*Similar calculations for odd n values (I=1) give
*
*ACCURACY=16  IND=I,1    METHOD=N
*ESTIMATE PART5,M5,CUR+1  / RESULTS=0  PRIND=0
*Estimated parameters of model M5:
*c0=0.467133970232680 (0.008400413530420)
*c1=-0.427082658486711 (0.000001661811032)
*c2=0.107638888841262 (0.000000000096496)
*c3=-0.010416666666666 (0.000000000000002)
*c4=0.000347222222222 (0.000000000000000)
*n=34993 rss=3439.353278 R^2=1.00000 nf=22
*
*1/c4=2880.0000000018435
```

```
*2880*c3=-29.999999999998082
*2880*c2=309.99999986283456
*2880*c1=-1229.9980564417276
*Preliminary suggestion is in this case
*Q(n,5)=n*(-1230+n*(310+n*(-30+n)))/2880 + eps  if mod(n,2)=1
*.....................................................................
*Studying residual eps for even n values
*
*Difference multiplied by 2880 between true and estimated values
*is computed:
*c4=1 c3=-30 c2=310 c1=-1320
*VAR q:8=2880*a-n*(c1+n*(c2+n*(c3+n*c4))) TO PART5
*.....................................................................
*Listing a part of the data with q values:
*
*FILE LOAD PART5 / SELECT=A*B A=I,0 B=n,16,132
*DATA PART5*,A,B,C
C          n                a   I         q
A         16                1   0      1984
*         18                3   0      1944
*         20                7   0      2560
*         22               13   0      1624
*         24               23   0      2304
*         26               37   0      1624
*         28               57   0      1984
*         30               84   0      2520
*         32              119   0      1984
*         34              164   0      1624
*         36              221   0      2304
*         38              291   0      1624
*         40              377   0      2560
*         42              480   0      1944
*         44              603   0      1984
*         46              748   0      1624
*         48              918   0      2304
*         50             1115   0      2200
*         52             1342   0      1984
*         54             1602   0      1944
*         56             1898   0      1984
*         58             2233   0      1624
*         60             2611   0      2880  !
*         62             3034   0      1624
*         64             3507   0      1984
*         66             4033   0      1944
*         68             4616   0      1984
*         70             5260   0      2200
*         72             5969   0      2304
*         74             6747   0      1624
*         76             7599   0      1984
*         78             8529   0      1944
*         80             9542   0      2560
*         82            10642   0      1624
*         84            11835   0      2304
*         86            13125   0      1624
*         88            14518   0      1984
*         90            16019   0      2520
*         92            17633   0      1984
*         94            19366   0      1624
```

```
*          96             21224   0       2304
*          98             23212   0       1624
*         100             25337   0       2560
*         102             27604   0       1944
*         104             30020   0       1984
*         106             32591   0       1624
*         108             35324   0       2304
*         110             38225   0       2200
*         112             41301   0       1984
*         114             44559   0       1944
*         116             48006   0       1984
*         118             51649   0       1624
*         120             55496   0       2880  !
*         122             59553   0       1624
*         124             63829   0       1984
*         126             68331   0       1944
*         128             73067   0       1984
*         130             78045   0       2200
B         132             83273   0       2304
*..................................................................
*The list gives a hunch that the residuals have a period of 60.
*This impression becomes still more convincing by computing 60-step
*differences of q values for n=61,62,...,10000:
*VAR diff=q-q[-60] TO PART5 / SELECT=A*B A=I,0 B=ORDER,61,10000
*..................................................................
*STAT PART5,CUR+1 / VARS=diff SELECT=A*B A=I,0 B=ORDER,61,10000
*Basic statistics: PART5 N=4970
*Variable: diff     ~q-q[-60]
*Constant= 0
*
*The results of the STAT command together with the table above makes it
*plausible that the q values have a period 60.
*..................................................................
*Saving the observed period obtained for even n values
*as a vector Q5 with 30 integer elements:
*IND=n,60,119  VARS=q
*MAT SAVE DATA PART5 TO A
*MAT A=VEC(A,2)     / *A~VEC(A) 2*30
*MAT Q5=A(1,*)'
*
*MAT LOAD Q5
*MATRIX Q5
*A(1,*)'
*///            1
*  1         2880
*  2         1624
*  3         1984
*  4         1944
*  5         1984
*  6         2200
*  7         2304
*  8         1624
*  9         1984
* 10         1944
* 11         2560
* 12         1624
* 13         2304
* 14         1624
```

```
* 15          1984
* 16          2520
* 17          1984
* 18          1624
* 19          2304
* 20          1624
* 21          2560
* 22          1944
* 23          1984
* 24          1624
* 25          2304
* 26          2200
* 27          1984
* 28          1944
* 29          1984
* 30          1624
*...................................................................
*Thus for even values of n, Q(n,5) seems to have an expression
*
*(S0) Q0_5(n):=(n*(n^3-30*n^2+310*n-1320)+MAT_Q5(mod(n/2,30)+1))/2880
*
*where according to notations of editorial computing in Survo
*MAT_Q5(i) denotes the i'th element of vector Q5
*and, for example,
*
*ACCURACY=16
*DAT_PARTS5(10000,a)=3461816314862   (correct value from data PARTS5)
*        Q0_5(10000)=3461816314862
*
*A similar study about the residuals q when n is odd leads to
*a suggestion
*
*(S1) Q1_5(n):=(n*(n^3-30*n^2+310*n-1230)+MAT_Q5(mod(mod((n+1)/2,30)+22,30)+1)-675)/2880
*
*and, for example,
*DAT_PARTS5(10001,a)=3463202289250
*        Q1_5(10001)=3463202289250
*
*Since all elements of the Q5 matrix are located in the closed interval
*[1624,2880], all residuals eps are in the interval
*[(1624-675)/2880,1]=[0.329513888...,1].
*
*Therefore by  using the functions
*floor(x):=int(x) and
*ceil(x):=if(int(x)=x)then(x)else(int(x)+1)
*the final formula for Q(n,5) is
*
*Q5(n):=ceil((n*(n^3-30*n^2+310*n+b5(n))+1)/2880)
*where
*b5(n):=if(mod(n,2)=0)then(-1320)else(-1230)
*
*This representation can also be transformed easily to form
*
*Q5(n):=ceil((n*(n^3-30*n^2+400*n-1320-180*floor(n/2))+1)/2880).
*
*...................................................................
*Proof of formulas (S0) and (S1)
*
```

```
*According to the previous calculations the formulas are valid
*at least for n=15,16,...,10000.
*It remains to show that these functions satisfy
*the recurrence relation (10) generally.
*Since there are 2*30=60 different forms (i.e. sets of polynomial coefficients)
*of Q(n,5) depending only on the value mod(n,60), it is sufficient to show
*their validity for 60 successive values of n by using (10), say for
*n=60,61,...,119.
*This is done safely in integer arithmetics by proving that F(n)=2880*Q(n,5)
*satisfies
*(F) F(n)=2880+F(n-2)+F(n-3)+F(n-4)-F(n-6)-2*F(n-7)-F(n-8)
*             +F(n-10)+F(n-11)+F(n-12)-F(n-14)
*
*The computational proof takes place by using two sucros (Survo macros).
*
*The first one POL1 simply creates F(n) for any value of n as a vector R.
*
*TUTSAVE POL1 / saving the sucro code on the following lines
*{tempo -1}
*{ins line}{ins line}{ins line}{ins line}{ins line}{ins line}{ins line}
*{ins line}{ins line}
*{u8}{line start}MATRIX R ///{R}
*{erase}mod({print W1},2)={act}{l} {save word W2}{line start}{erase}
- if W2 = 1 then goto A
*{erase}MAT_Q5(mod({print W1}/2,30)+1){R}
*-1320{R}
*310{R}
*-30{R}
*1{R}
*{erase}{goto B}
+ A:
*{erase}MAT_Q5(mod(mod(({print W1}+1)/2,30)+22,30)+1)-675{R}
*-1230{R}
*310{R}
*-30{R}
*1{R}
*{erase}
/
+ B:
*{R}
*MAT SAVE R{act}{R}
*MAT RLABELS NUM(0) TO R{act}{R}{u9}
*{del line}{del line}{del line}{del line}{del line}{del line}{del line}
*{del line}{del line}{u}
*{end}
*
*Example:
*/POL1 60
*MAT LOAD R'
*MATRIX R'
*///            0        1        2        3        4
*  1          2880    -1320      310      -30        1
*
*The second sucro POL2 computes the coefficients of polynomial
*obtained as the right-side expression of (F) as a vector P
*and F(n) as a vector R. Both vectors are displayed in the edit field
*for a comparison.
*
```

```
*TUTSAVE POL2 / saving the sucro code on the following lines
*{tempo -1}{R}
/
*MAT P=P0{act}{R}
*{W2=W1-2}
*{save stack}/POL1 {print W2}{act}{del stack}{load stack}{R}
*POL R=LAG(R,2){act}{R}
*POL P=P+R{act}{R}
*{W2=W1-3}
*{save stack}/POL1 {print W2}{act}{del stack}{load stack}{R}
*POL R=LAG(R,3){act}{R}
*POL P=P+R{act}{R}
*{W2=W1-4}
*{save stack}/POL1 {print W2}{act}{del stack}{load stack}{R}
*POL R=LAG(R,4){act}{R}
*POL P=P+R{act}{R}
*{W2=W1-6}
*{save stack}/POL1 {print W2}{act}{del stack}{load stack}{R}
*POL R=LAG(R,6){act}{R}
*POL P=P-R{act}{R}
*{W2=W1-7}
*{save stack}/POL1 {print W2}{act}{del stack}{load stack}{R}
*POL R=LAG(R,7){act}{R}
*POL P=P-R{act}{R}
*POL P=P-R{act}{R}
*{W2=W1-8}
*{save stack}/POL1 {print W2}{act}{del stack}{load stack}{R}
*POL R=LAG(R,8){act}{R}
*POL P=P-R{act}{R}
*{W2=W1-10}
*{save stack}/POL1 {print W2}{act}{del stack}{load stack}{R}
*POL R=LAG(R,10){act}{R}
*POL P=P+R{act}{R}
*{W2=W1-11}
*{save stack}/POL1 {print W2}{act}{del stack}{load stack}{R}
*POL R=LAG(R,11){act}{R}
*POL P=P+R{act}{R}
*{W2=W1-12}
*{save stack}/POL1 {print W2}{act}{del stack}{load stack}{R}
*POL R=LAG(R,12){act}{R}
*POL P=P+R{act}{R}
*{W2=W1-14}
*{save stack}/POL1 {print W2}{act}{del stack}{load stack}{R}
*POL R=LAG(R,14){act}{R}
*POL P=P-R{act}{R}
/
*/POL1 {print W1}{act}{R}
*MAT LOAD P'{act}{R}
*{d5}
*MAT LOAD R'{act}{R}
*{end}
*
*For example, POL R=LAG(R,6) substitutes n in R(n) by n-6
*and expands R(n-6) to a polynomial in powers of n.
*
*The starting polynomial corresponding to constant 2880 is
*given as a vector P0:
*
```

```
*MATRIX P0
*///          real
*  0          2880
*  1             0
*  2             0
*  3             0
*  4             0
*
*MAT SAVE P0
*
*When /POL2 is activated with the argument 60, it gives
*/POL2  60
*MAT P=P0
*/POL1 58
*POL R=LAG(R,2)
*POL P=P+R
*/POL1 57
*POL R=LAG(R,3)
*POL P=P+R
*/POL1 56
*POL R=LAG(R,4)
*POL P=P+R
*/POL1 54
*POL R=LAG(R,6)
*POL P=P-R
*/POL1 53
*POL R=LAG(R,7)
*POL P=P-R
*POL P=P-R
*/POL1 52
*POL R=LAG(R,8)
*POL P=P-R
*/POL1 50
*POL R=LAG(R,10)
*POL P=P+R
*/POL1 49
*POL R=LAG(R,11)
*POL P=P+R
*/POL1 48
*POL R=LAG(R,12)
*POL P=P+R
*/POL1 46
*POL R=LAG(R,14)
*POL P=P-R
*/POL1 60
*MAT LOAD P'
*MATRIX P'
*Polynom'
*///            0        1        2        3        4
*real          2880    -1320      310      -30        1
*
*MAT LOAD R'
*MATRIX R'
*///            0        1        2        3        4
*  1          2880    -1320      310      -30        1
*
*and we see that the polynomials are identical.
*By activating
```

```
*/POL2 61
*we get
*- - -
*MAT LOAD P'
*MATRIX P'
*Polynom'
*///            0       1       2       3       4
*real         949   -1230     310     -30       1
*
*MAT LOAD R'
*MATRIX R'
*///            0       1       2       3       4
*  1          949   -1230     310     -30       1
*
*and so on until
*/POL2 119
*giving
*- - -
*MAT LOAD P'
*MATRIX P'
*Polynom'
*///            0       1       2       3       4
*real        1309   -1230     310     -30       1
*
*MAT LOAD R'
*MATRIX R'
*///            0       1       2       3       4
*  1         1309   -1230     310     -30       1
*
*and it becomes clear that in each case n=60,61,...,119
*the P and R polynomials are identical.
*This proves that (F) and (10) are valid for any n>=15.
-------------------------------------------------------------------
```

A similar procedure leads to the following results:

(11) $Q(n,2) = \lfloor (n-1)/2 \rfloor$,

(12) $Q(n,3) = \lceil (n(n-6)+1)/12 \rceil$,

(13) $Q(n,4) = \lfloor (n(n^2 - 24n + 72 + 18\lfloor n/2 \rfloor) - 1)/144 \rfloor$

or alternatively

(14) $Q(n,4) = \lfloor (n(n^2 - 15n + b_4(n)) - 1)/144 \rfloor$

where

$b_4(n) = 72$ if n is even and $b_4(n) = 63$ if n is odd,

(15) $Q(n,5) = \lceil (n(n^3 - 30n^2 + 400n - 1320 - 180\lfloor n/2 \rfloor) + 1)/2880 \rceil$

or alternatively

(16) $Q(n,5) = \lceil (n(n^3 - 30n^2 + 310n - b_5(n)) + 1)/2880 \rceil$

where

$b_5(n) = 1320$ if n is even and $b_5(n) = 1230$ if n is odd,

(17) $Q(n,6) = \lfloor (n(6n^4 - 315n^3 + 6160n^2$

$- \mathbf{b}_6[\mathrm{mod}(n,2)+1]n + \mathbf{c}_6[\mathrm{mod}(n,6)+1]) - 1)/518400 \rfloor$

where

$\mathbf{b}_6 = (55800, 54450)$,

$\mathbf{c}_6 = (240480, 202530, 230880, 212130, 230880, 202530)$,

(18) $Q(n,7) = \lceil (n(n^5 - 84n^4 + 2765n^3 - 45080n^2$

$+ \mathbf{b}_7[\mathrm{mod}(n,2)+1]n - \mathbf{c}_7[\mathrm{mod}(n,6)+1]) + 1)/3628800 \rceil$

where

$\mathbf{b}_7 = (379512, 374787)$,

$\mathbf{c}_7 = (1582560, 1405460, 1560160, 1450260, 1537760, 1427860)$.

The leading term in all $Q(n,m)$ expressions above is

$$\frac{n^{m-1}}{m[(m-1)!]^2}$$

Assuming that in general $Q(n,m) = cn^{m-1} + O(n^{m-2})$ it is easy to prove by induction and by using the recursion formula (6) that

$$Q(n,m) \approx \frac{n^{m-1}}{m[(m-1)!]^2}. \tag{19}$$

For example, for $Q(10000,6)$ this approximation deviates only about 0.5 per cent from the exact value.

The recursion formula (6) is also a useful tool for deriving exact formulas for even greater m values.

References

[1] Andrews, G. E. and Eriksson, K., *Integer Partitions*, Cambridge University Press, 2004.
[2] Kreher, D.L and Stinson, D. R., *Combinatorial Algorithms*, CRC Press, 1999.
[3] `http://www.survo.fi/english`