EXTENSION OF GOLDEN SECTION TO MULTIPLE-PARTITE DIVISION OF A LINE SEGMENT

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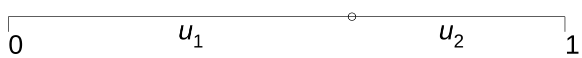
In this note we generalize the classical Golden Section presented originally by Euclid [1] in his book VI by

Definition 3:
A straight line is said to have been cut in *extreme and mean ratio* when, as the whole line is to the greater segment, so is the greater to the less.

The name Golden Section or Golden Ratio has been introduced much later in early 1800's and also a term *division in continuous proportion* has been used since then, see e.g. [2].

The Golden Ratio ϕ is obtained by studying a unit line segment



and dividing it into two parts of lengths u_1, u_2 so that

$$u_1 + u_2 = 1 \tag{1}$$

and

$$1/u_1 = u_1/u_2 = \phi \tag{2}$$

according to Euclid's definition. From (1) and (2) we get for ϕ an equation of second degree

$$\phi^2 - \phi - 1 = 0 \tag{3}$$

which gives the value $\phi = (\sqrt{5} + 1)/2 = 1.618...$

It is a well-known fact that the same constant ϕ is obtained as the Fibonacci constant $lim_{k\to\infty}(F_k/F_{k-1})$ of the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

where

$$F_k = F_{k-1} + F_{k-2},\ k = 3, 4, \ldots \tag{4}$$

since simply by writing (4) in the form

$$\frac{F_n}{F_{n-1}} \frac{F_{n-1}}{F_{n-2}} = \frac{F_{n-1}}{F_{n-2}} + 1$$

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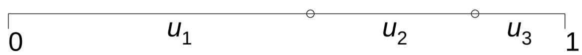
we see that the limiting ratio satisfies the equation (3).

Now we extend the idea of *division in continuous proportion* in the simplest possible way by dividing the unit line segment



into three parts

$$u_1 + u_2 + u_3 = 1 \tag{5}$$

so that 'as the whole line is to the greatest segment, so is the greatest to the midmost, and so is the midmost is to the smallest', i.e.

$$1/u_1 = u_1/u_2 = u_2/u_3 = \phi_3. \tag{6}$$

The requirements (5),(6) lead to an equation of third degree for the common ratio ϕ_3

$$\phi_3^3 - \phi_3^2 - \phi_3 - 1 = 0 \tag{7}$$

having only one real root

$$\phi_3 = \frac{1}{3}(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}) = 1.839286755214\ldots$$

It is not a big surprise that ϕ_3 equals to the Tribonacci constant [3] which is obtained as the limiting ratio of consecutive terms in the Tribonacci sequence

$$1, 1, 2, 4, 7, 13, 24, 44, 81, \ldots,$$

where

$$F_k^{(3)} = F_{k-1}^{(3)} + F_{k-2}^{(3)} + F_{k-3}^{(3)},\ k = 4, 5, \ldots \tag{8}$$

since now $lim_{k\to\infty}(F_k^{(3)}/F_{k-1}^{(3)})$ is the only real root of equation (7). However, a bigger surprise is that our simple geometric generalization of the Golden Section seems to be neglected in earlier literature.

In general, we may study division of a unit line segment in n parts

$$u_1 + u_2 + \cdots + u_n = 1,\ n = 1, 2, 3, \ldots \tag{9}$$

and in continuous proportion(s)

$$1/u_1 = u_1/u_2 = u_2/u_3 = \cdots = u_{n-1}/u_n = \phi_n. \tag{10}$$

Then it is easy to see that ϕ_n is the only real positive root of

$$\phi_n^n - \phi_n^{n-1} - \phi_n^{n-2} - \cdots - \phi_n - 1 = 0. \tag{11}$$

The same constant is obtained as the limiting ratio two consecutive terms of Fibonacci n-step numbers [4] since this ratio satifies (11).

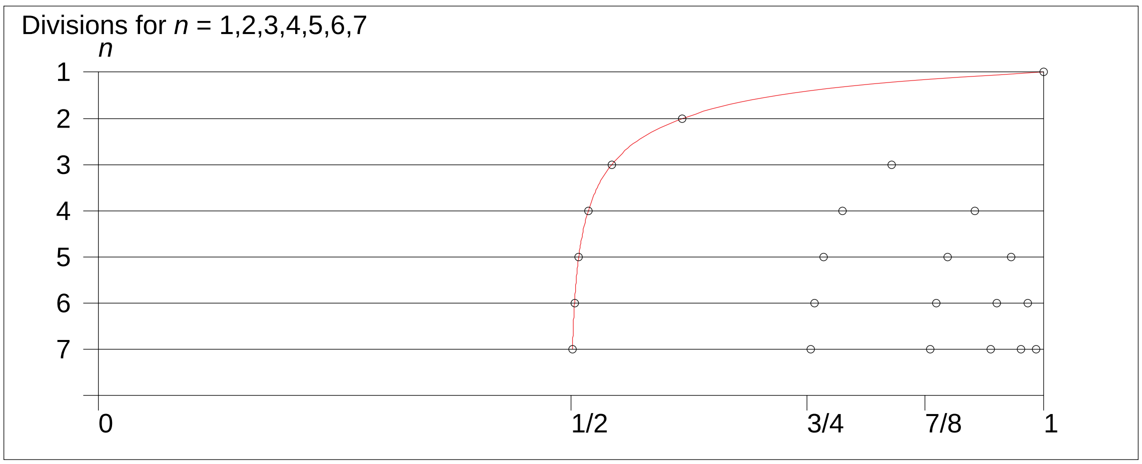
According to our geometric approach it is clear that ϕ_n approaches 2 as $n \to \infty$ and the n-partite division tends to an infinite bipartition

$$u_1 = 1/2, u_2 = 1/4, u_3 = 1/8, \ldots, u_i = (1/2)^i, \ldots$$

As special cases we have $\phi_1 = 1, \phi_2 = \phi$, and the ϕ-values can be summarized as

ratio	name	value	Sloane [5]
ϕ_1	1	1	
ϕ_2	Golden Ratio	1.618033988 ...	A001622
ϕ_3	Tribonacci constant	1.839286755 ...	A058265
ϕ_4	Tetranacci constant	1.927561975 ...	A086088
ϕ_5	Pentanacci constant	1.965948236 ...	A103814
ϕ_6	Hexanacci constant	1.983582843 ...	
ϕ_7	Heptanacci constant	1.991964196 ...	
...			
ϕ_∞	2	2	

and the corresponding divisions as



The red curve is based on an approximation

$$\phi_n \approx 2 - \frac{1}{2}[2(1+2/\sqrt{5})]^{-n+1} - 2^{-n}.$$

The current version of this paper can be downloded from
`http://www.survo.fi/papers/nsection.pdf`

References

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