LOGARITHMIC MEAN FOR SEVERAL ARGUMENTS

SEPPO MUSTONEN

Abstract. The logarithmic mean is generalized for n positive arguments $x_1, \ldots, x_n$ by examining series expansions of typical mean numbers in case $n = 2$. The generalized logarithmic mean defined as a series expansion can then be presented also in closed form which proves to be the $(n-1)$th divided difference (multiplied by $(n-1)!$) of values $f(u_1), \ldots, f(u_n)$ where $f(u_i) = e^{u_i} = x_i$, $i = 1, \ldots, n$. Various properties of this generalization are studied and an efficient recursive algorithm for computing it is presented.

1. Introduction

Some statisticians and mathematicians have proposed generalizations of the logarithmic mean for n arguments ($n > 2$), see E.L.Dodd [3] and A.O.Pittenger [11].

The generalization presented in this paper differs from the earlier suggestions and has its origin in an unpublished manuscript of the author [6]. This manuscript based on a research made in early 70's is referred to in the paper of L.Törnqvist, P.Vartia, Y.O.Vartia [13]. It essentially described a generalization in cases $n = 3, 4$ and provided a suggestion for a general form which will be derived in this paper.

The logarithmic mean $L(x_1, x_2)$ for two arguments $x_1 > 0$, $x_2 > 0$ is defined by

$$L(x_1, x_2) = \frac{x_1 - x_2}{\log (x_1/x_2)} \text{ for } x_1 \neq x_2 \text{ and } L(x_1, x_1) = x_1. \tag{1}$$

Obviously Leo Törnqvist was the first to advance the "log-mean" concept in his fundamental research work related to price indexes [12]. Yrjö Vartia then implemented the logarithmic mean in his log-change index numbers [14].

In [13] the log-change $\log(x_2/x_1)$ is suggested to be used instead of the common relative change $(x_2 - x_1)/x_1$ as an indicator of relative change for several theoretical and practical reasons. It is connected to the logarithmic mean simply by

$$\log (x_2/x_1) = \frac{x_1 - x_2}{L(x_1, x_2)}. \tag{2}$$

Among other things it will be shown that a corresponding formula is valid in the generalized case.

2. Generalization

The starting point for the generalization is the observation that $L(x_1, x_2)$ is found to be related to the arithmetic mean $A(x_1, x_2) = (x_1 + x_2)/2$ and the geometric mean $G(x_1, x_2) = \sqrt{x_1 x_2}$ by using suitable series expansions for each of them.

Date: October 7, 2002, revised October 12, 2002, Appendix 1: December 26, 2002, Appendix 2 (by J. Merikoski): June 10, 2003, Appendix 3 (by J. Merikoski): October 7, 2003.

Key words and phrases. logarithmic mean, series expansions, divided differences.

I would like to thank Yrjö Vartia for his inspiring interest in my attempts in this project from early 1970's and Jorma Merikoski for a valuable remark related to divided differences.

By denoting

$$x_1 = \exp u_1,\ \ x_2 = \exp u_2$$

the following expansions based on

$$\exp u = 1 + u + u^2/2! + u^3/3! + \dots$$

are immediately obtained:

$$\begin{aligned}
A(x_1, x_2) &= 1 + (u_1 + u_2)/2 + (u_1^2 + u_2^2)/(2 \cdot 2!) + (u_1^3 + u_2^3)/(2 \cdot 3!) + \dots, \\
G(x_1, x_2) &= \sqrt{e^{u_1} e^{u_2}} = \exp\left[(u_1 + u_2)/2\right] \\
&= 1 + (u_1 + u_2)/2 + (u_1 + u_2)^2/(2^2 \cdot 2!) + (u_1 + u_2)^3/(2^3 \cdot 3!) + \dots \\
&= 1 + (u_1 + u_2)/2 + (u_1^2 + 2u_1u_2 + u_2^2)/(2^2 \cdot 2!) \\
&\qquad\qquad + (u_1^3 + 3u_1^2u_2 + 3u_1u_2^2 + u_2^3)/(2^3 \cdot 3!) + \dots, \\
L(x_1, x_2) &= (e^{u_1} - e^{u_2})/(u_1 - u_2) \\
&= 1 + (u_1 + u_2)/2 + (u_1^2 + u_1u_2 + u_2^2)/(3 \cdot 2!) \\
&\qquad\qquad + (u_1^3 + u_1^2u_2 + u_1u_2^2 + u_2^3)/(4 \cdot 3!) + \dots.
\end{aligned}$$

The expansions are identical up to the first degree. In the term of degree $m > 1$ the essential factor is a symmetric homogeneous polynomial of the form

$$B_m u_1^m + B_{m-1} u_1^{m-1} u_2 + B_{m-2} u_1^{m-2} u_2^2 + \dots + B_0 u_2^m$$

divided by the sum of its coefficients $B_m, B_{m-1}, \dots, B_0$. These coefficients characterize each of the means completely.

In the arithmetic mean we have

$$B_0 = B_1 = 1 \text{ and } B_2 = \dots = B_{m-1} = 0.$$

In the geometric mean they are binomial coefficients

$$B_i = C(m, i),\ i = 0, 1, \dots, m$$

and in the logarithmic mean all coefficients equal to 1:

$$B_i = 1,\ i = 0, 1, \dots, m.$$

The coefficients of the logarithmic mean arise from division $(u_1^{m+1} - u_2^{m+1})/(u_1 - u_2)$ which symmetrizes its structure. Also other means (like harmonic and moment means) have similar expansions but their B coefficients are more complicated. The logarithmic mean has the most balanced B structure.

On the basis of this fact it was natural to generalize L in such a way that it keeps this simple structure. Thus the logarithmic mean for n observations

$$x_i = \exp u_i,\ i = 1, 2, \dots, n$$

is defined by

$$
\begin{aligned}
L(x_1,x_2,\dots,x_n) = 1 \; &+ (u_1+u_2+\cdots+u_n)/n \\
&+ \frac{u_1^2+u_1u_2+\cdots+u_1u_n+u_2^2+u_2u_3+\cdots+u_n^2}{C(n+1,2)\cdot 2!} \\
&+ \dots \\
&+ \frac{u_1^m+u_1^{m-1}u_2+\cdots+u_n^m}{C(n+m-1,m)\cdot m!} \\
&+ \dots .
\end{aligned} \tag{3}
$$

In this series expansion the polynomial in the term of degree m has the form

$$P(n,m) = \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ i_1\geq 0, i_2\geq 0,\dots,i_n\geq 0}} u_1^{i_1}u_2^{i_2}\dots u_n^{i_n}$$

and so the all B coefficients are equal to 1. They have divisors $C(n+m-1,m)$ corresponding to the number of summands.

In my earlier study [6] I succeeded in transforming this expansion to a closed form

$$L(x_1,x_2,\dots,x_n) = (n-1)!\sum_{i=1}^{n}\frac{x_i}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}\log\left(x_i/x_j\right)} \tag{4}$$

when all the x's are mutually different positive numbers. In fact, I was then able to prove (4) in cases $n=3,4$ and the general form was only a natural conjecture. I lost my interest in further studies since the formula is numerically very unstable for large n values. It is better to use the series expansion (3) in practice. However, in theoretical considerations (4) is important.

3. Derivation of the formula (4)

Polynomials $P(n,m)$ can be represented in a recursive form according to decreasing powers of the last u as

$$
\begin{aligned}
P(n,m) = \quad & u_n^m \\
&+ u_n^{m-1}P(n-1,1) \\
&+ u_n^{m-2}P(n-1,2) \\
&\quad\dots \\
&+ u_n^{1}P(n-1,m-1) \\
&+ u_n^{0}P(n-1,m)
\end{aligned} \tag{5}
$$

with side conditions $P(n,1)=u_1+u_2+\cdots+u_n$, $P(1,m)=u_1^m$.

If all x's (and therefore also u's) are mutually different, it is fundamental to notice that polynomials $P(n,m)$ can be represented by another way by using expressions

$$Q(n,m) = \sum_{i=1}^{n}\frac{u_i^m}{U_i},\; m=0,1,2,\dots \tag{6}$$

where

$$U_i = \prod_{\substack{j=1 \\ j\neq i}}^{n} (u_i - u_j),\ i = 1, 2, \ldots, n. \tag{7}$$

The following identities are valid and will be proved in the next chapter.

$$Q(n, m) = 0 \text{ for } m = 0, 1, 2, \ldots, n-2, \tag{8}$$

$$Q(n, n-1) = 1, \tag{9}$$

$$Q(n, m) = P(n, m-n+1) \text{ for } m = n, n+1, n+2, \ldots\ . \tag{10}$$

By means of these identities the formula (4) can be derived from the definition (3) as follows:

$$\begin{aligned}
L(x_1, x_2, \ldots, x_n) &= 1 + P(n,1)/n + P(n,2)/[C(n+1,2)\cdot 2!] + \ldots \\
&\quad + P(n,m)/[C(n+m-1,m)\cdot m!] + \ldots \\
&= 1 + (n-1)! \sum_{m=1}^{\infty} \frac{P(n,m)}{(n+m-1)!} \\
&= 1 + (n-1)! \sum_{m=1}^{\infty} \frac{Q(n,n+m-1)}{(n+m-1)!} \quad \text{from (10)} \\
&= 1 + (n-1)! \sum_{k=n}^{\infty} \frac{Q(n,k)}{k!} \\
&= (n-1)! \sum_{k=n-1}^{\infty} \frac{Q(n,k)}{k!} \quad \text{from (9)} \\
&= (n-1)! \sum_{k=0}^{\infty} \frac{Q(n,k)}{k!} \quad \text{from (8)} \\
&= (n-1)! \sum_{k=0}^{\infty} \frac{\sum_{i=1}^{n} u_i^k / U_i}{k!} \quad \text{from (6)} \\
&= (n-1)! \sum_{i=1}^{n} \frac{\sum_{k=0}^{\infty} u_i^k / k!}{U_i} \\
&= (n-1)! \sum_{i=1}^{n} \frac{\exp u_i}{\prod_{\substack{j=1 \\ j\neq i}}^{n} (u_i - u_j)} \quad \text{from (7)}
\end{aligned}$$

which is identical with (4) since $u_i = \log x_i,\ i = 1, 2, \ldots, n$.

4. Proof of identities (8), (9), (10)

It can be seen immediately that the identities are valid for $n = 2$. In this case

$$Q(2,k) = u_1^k/(u_1 - u_2) + u_2^k/(u_2 - u_1) = (u_1^k - u_2^k)/(u_1 - u_2),\ k = 0, 1, 2, \ldots$$

and thus

$$Q(2,0) = 0,\ Q(2,1) = 1 \text{ and } Q(2,k) = P(2,k-1) \text{ for } k = 2, 3, \ldots\ .$$

The general proof is based on induction from $n-1$ to n. Thus by assuming that the identities are valid in case $n-1$ it will be shown that they are valid in case n, too.

By writing denominators u_i^m of (6) in the form $(u_i^m - u_n^m) + u_n^m$ and by splitting these terms and by dividing the first part by the last factor $u_i - u_n$ in divisor (7) we get a recursion formula

$$\begin{aligned} Q(n,m) = &\quad u_n^{m-1} Q(n-1,0) \\ &+ u_n^{m-2} Q(n-1,1) \\ &\quad \dots \\ &+ u_n^0 Q(n-1,m-1) + u_n^m Q(n,0),\ m = 1,2,\dots\ . \end{aligned} \tag{11}$$

Let us denote $Q(n,0) = f(u_1, u_2, \dots, u_n)$ and study the function f with the inverse values of its arguments, i.e. the function $f(1/u_1, 1/u_2, \dots, 1/u_n)$. Then the expressions $1/u_i - 1/u_j$ can be written in the form $(u_j - u_i)/(u_i u_j)$ and after simplification we get

$$f(1/u_1, 1/u_2, \dots, 1/u_n) = (-1)^n u_1 u_2 \dots u_n Q(n, n-2).$$

By applying the recursion formula (11) to the last factor and by observing that (8) is valid in case $n-1$, we see that only the last term in the recursion formula can be different from 0 and hence

$$f(1/u_1, 1/u_2, \dots, 1/u_n) = (-1)^n u_1 u_2 \dots u_n u_n^{n-2} f(u_1, u_2, \dots, u_n).$$

Function $f(u_1, u_2, \dots, u_n)$ is homogeneous and symmetric. If f were else than identically zero, it leads to a contradiction since the right side of the last equation could not be a symmetric function in cases $n > 2$. Thus $Q(n,0) = 0$ for $n = 2, 3, \dots$ and (8) has been proved in case $m = 0$.

Then in (11) the last term can be omitted and we have

$$\begin{aligned} Q(n,m) = &\quad u_n^{m-1} Q(n-1,0) \\ &+ u_n^{m-2} Q(n-1,1) \\ &\quad \dots \\ &+ u_n^0 Q(n-1,m-1),\ m = 1,2,\dots\ . \end{aligned} \tag{12}$$

By the induction assumption this gives

$$\begin{aligned} Q(n,1) &= u_n^0 Q(n-1,0) = 0, \\ Q(n,2) &= u_n^1 Q(n-1,0) + u_n^0 Q(n-1,1) = 0, \\ &\dots \\ Q(n,n-2) &= u_n^{n-3} Q(n-1,0) + \dots + u_n^0 Q(n-1,n-3) = 0 \end{aligned}$$

and so (8) has been proved also for $m = 1, 2, \dots, n-2$.

In case $m = n-1$ (12) gives

$$Q(n,n-1) = u_n^0 Q(n-1,n-2) = 1$$

and (9) is valid.

In case $m = n$ (12) gives

$$\begin{aligned} Q(n,n) &= u_n^1 Q(n-1,n-2) + u_n^0 Q(n-1,n-1) \\ &= u_n + (u_1 + u_2 + \dots + u_{n-1}) = u_1 + u_2 + \dots + u_n \end{aligned}$$

and (10) is valid when $m = n$ and hence $Q(n, n) = P(n, 1)$.

By these results the recursion formula (12) is reduced to the form

$$
\begin{aligned}
Q(n,m) = {} & \quad u_n^{m-n+1} \\
& + u_n^{m-n} Q(n-1, n-1) \\
& \quad \dots \\
& + u_n^0 Q(n-1, m-1), \ m = n, n+1, \dots \ .
\end{aligned} \tag{13}
$$

By using this formula and (10) for $n - 1$ we get

$$
\begin{aligned}
Q(n, n+1) &= u_n^2 + u_n^1 Q(n-1, n-1) + u_n^0 Q(n-1, n) \\
&= u_n^2 + u_n P(n-1, 1) + P(n-1, 2) \\
&= P(n, 2) \quad \text{ from (5)}
\end{aligned}
$$

which means that (10) is valid for $m = n+1$ and $Q(n, n+1) = P(n, 2)$. Similarly, when $m > n$ we obtain by using (13) and (10) (the latter for $n - 1$)

$$
\begin{aligned}
Q(n,m) = {} & \quad u_n^{m-n+1} \\
& + u_n^{m-n} P(n-1, 1) \\
& + u_n^{m-n-1} P(n-1, 2) \\
& \quad \dots \\
& + u_n^0 P(n-1, m-n+1) = P(n, m-n+1) \quad \text{ from (5)}
\end{aligned}
$$

and this proves (10) in general.

5. Logarithmic mean and divided differences

Since I felt that identities (8) and (9) must be known in some other connections and, in particular, the denominators (7) are present also in the Lagrange's interpolation formula, I sent an inquiry about their origin to some of my colleagues in Finland.

Jorma Merikoski (University of Tampere) remarked immediately that in fact (8) and (9) are well-known identities when considering divided differences (in the Lagrangian interpolation scheme) for powers u^k, $k = 0, 1, \dots, n-2$.

His note led me to find out that (4) is equal to the (only) $(n-1)$th order divided difference of function values $x_i = \exp u_i$, $i = 1, 2, \dots, n$, multiplied by $(n-1)!$ (See e.g. C.E.Fröberg [4] p. 148).

For example, in case $n = 3$ the divided differences are

u	$f(u)$	1st difference	2nd difference
u_1	$\exp u_1$		
		$\dfrac{\exp u_2 - \exp u_1}{u_2 - u_1}$	
u_2	$\exp u_2$		$\dfrac{\dfrac{\exp u_3 - \exp u_2}{u_3 - u_2} - \dfrac{\exp u_2 - \exp u_1}{u_2 - u_1}}{u_3 - u_2}$
		$\dfrac{\exp u_3 - \exp u_2}{u_3 - u_2}$	
u_3	$\exp u_3$		

and the second divided difference is equal to
$L(\exp u_1, \exp u_2, \exp u_3)/2 =$

$$\frac{\exp u_1}{(u_1 - u_2)(u_1 - u_3)} + \frac{\exp u_2}{(u_2 - u_1)(u_2 - u_3)} + \frac{\exp u_3}{(u_3 - u_1)(u_3 - u_2)}.$$

This means that $L(x_1, \ldots, x_n)$ can be computed recursively according to the formula

$$L(x_1, \ldots, x_n) = (n-1)\frac{L(x_2, \ldots, x_n) - L(x_1, \ldots, x_{n-1})}{\log(x_n/x_1)} \text{ for } n = 2, 3, \ldots \tag{14}$$

Since, according to the classical mean value theorem the $(n-1)$th divided difference $d(u_1, \ldots, u_n)$ for function values $f(u_1), \ldots, f(u_n)$ (for a function f which is continuously differentiable $n-1$ times) can represented in the form (see Fröberg [4], p. 148)

$$d(u_1, \ldots, u_n) = \frac{f^{(n-1)}(\xi)}{(n-1)!}$$

where $\min(u_1, \ldots, u_n) < \xi < \max(u_1, \ldots, u_n)$ we have now $f(u) = \exp u$ with all derivatives identically equal to $f(u)$ and hence

$$L(x_1, \ldots, x_n) = e^{\xi}.$$

Thus the logarithmic mean is directly related to a 'mean value' also in the sense of standard analysis for real functions.

6. Relative changes

By (14) the relative change $\log(x_n/x_1)$ can be written as

$$\log(x_n/x_1) = (n-1)\frac{L(x_2, \ldots, x_n) - L(x_1, \ldots, x_{n-1})}{L(x_1, \ldots, x_n)}.$$

Since trivially

$$\frac{x_n}{x_1} = \frac{x_2}{x_1} \cdot \frac{x_3}{x_2} \cdot \ \ldots \ \cdot \frac{x_n}{x_{n-1}},$$

we have

$$\frac{\sum_{i=1}^{n-1} \log(x_{i+1}/x_i)}{n-1} = \frac{L(x_2, \ldots, x_n) - L(x_1, \ldots, x_{n-1})}{L(x_1, \ldots, x_n)},$$

i.e. the average of the log-changes in series of observations $x_1, x_2, \ldots, x_n$ is equal to a natural generalization of the right-hand side in (2).

7. Logarithmic mean for exponentially growing data

Let us consider the data set

$$x_0, \ x_0 c, \ x_0 c^2, \ x_0 c^3, \ \ldots, \ x_0 c^{n-1}.$$

In this case (4) can be written in the form

$$L(x_1, \ldots, x_n) = \frac{(n-1)!\, x_0}{(\log c)^{n-1}} \sum_{i=1}^{n} \frac{c^{i-1}}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} (i-j)}.$$

The divisors in the sum are of the form $(-1)^{n-i}(i-1)!(n-i)!$ and then according to the formula $C(m,k) = m!/[k! * (m-k)!]$ for binomial coefficients we have

$$\begin{aligned} L(x_1,\ldots,x_n) &= \frac{(n-1)!\,x_0}{(\log c)^{n-1}} \sum_{i=1}^{n} \frac{(-1)^{n-i}C(n-1,i-1)c^{i-1}}{(n-1)!} \\ &= \frac{(n-1)!\,x_0}{(\log c)^{n-1}} \times \frac{(c-1)^{n-1}}{(n-1)!} \qquad \text{(from binomial formula)} \\ &= x_0[(c-1)/\log c]^{n-1} \\ &= x_0 L(c,1)^{n-1}. \end{aligned}$$

Thus when the observations are growing by a constant factor $c > 1$, the logarithmic mean grows by a constant factor $L(c,1)$. Apparently the same result is obtained for $0 < c < 1$, too.

In fact, a corresponding result is valid for the geometric mean since we get immediately that

$$G(x_1, x_2, \ldots, x_n) = x_0 G(c,1)^{n-1}$$

where $G(c,1) = \sqrt{c \cdot 1}$. It shows certain similarity between the geometric and logarithmic mean. However, when $c \neq 1$, it follows that $\lim_{n\to\infty} L/G = \infty$ since

$$L(c,1) > G(c,1). \tag{15}$$

Inequality (15) for $c > 1$ can be proved simply by studying the behaviour of the function $f(x) = \log x[L(x^2,1) - G(x^2,1)] = (x^2-1)/2 - x\log x$ for $x > 1$. Since

$$L(ax, ay) = aL(x,y) \text{ and } G(ax,ay) = aG(x,y) \text{ for } a > 0, \tag{16}$$

it follows immediately that (15) is valid also for $0 < c < 1$. Hence (15) has been proved for all positive $c \neq 1$. Similarly the inequality $L(x,y) > G(x,y)$ for $x \neq y$ is proved by using (15) and (16). Of course, other general proofs are available, see e.g. B.C.Carlson [1].

8. Computational aspects

In principle, the generalized logarithmic mean can be computed quickly from the closed form (4) but this fails numerically for $n > 14$ although double precision is used. The reason for this unpleasant phenomen is the fact that (4) is a sum of 'huge' alternating terms and the number of significant digits are soon lost. Furthermore (4) is not applicable at all when some x's are equal. Also the recursive formula (14) suffers for same reasons.

Hence the main method for computing logarithmic means in the statistical system Survo (Mustonen [7], `http://www.survo.fi`) is based on the original definition i.e. the series expansion (3). For this task I have created a new Survo program module `LOGMEAN`.

When using the series expansion it is essential how the symmetric, homogeneous polynomials $P(m,n)$ are evaluated. It is done by using the recursive formula (5). To speed up the recursion process the `LOGMEAN` module saves all computed $P(n,m)$ values in a table. Thus in each recursive step it is checked whether the current $P(n,m)$ has been already calculated. By this technique cases where n is less than 10000 are calculated very rapidly but on current PC's also cases where n is much higher can be handled.

For example, for a data set 1, 2, 3, ... , n ($n = 200000$) `LOGMEAN` gives

$$L_n = 73578.65538616560 \quad \text{(logarithmic mean)}$$
$$G_n = 73578.47151997556 \quad \text{(geometric mean)}$$

and after doing the same when the last observation 200000 is omitted we get for $n = 200000$

$$L_n - L_{n-1} = 0.36788036154758 \quad L_n/n = 0.36789327693083$$
$$G_n - G_{n-1} = 0.36788036060170 \quad G_n/n = 0.36789235759988$$

On the basis of these calculations it is obvious that

$$\lim_{n\to\infty}(L_n - L_{n-1}) = \lim_{n\to\infty}(G_n - G_{n-1}) = 1/e = 0.367879\ldots$$

and also

$$\lim_{n\to\infty}(L_n/n) = \lim_{n\to\infty}(G_n/n) = 1/e.$$

For the geometric mean these results can be proved by Stirling's formula. The same is not yet proved for the logarithmic mean.

9. Concluding remarks

The generalization presented in this paper comes close to that of Pittenger [11] in certain aspects. However, already numerical examples with $n = 3$ show that it these generalizations are not the same. Also in principle Pittenger's approach is different since he starts from the inverse of $L(x_1, x_2)$ and by following Carlson [1] writes this inverse as a certain definite integral which is then extended into multivariable form and finally represented as a closed expression.

It is obvious that the generalized logarithmic mean as defined in this paper satisfies inequalities

$$G(x_1, \ldots, x_n) \le L(x_1, \ldots, x_n) \le A(x_1, \ldots, x_n) \tag{17}$$

but it has not been proved for $n > 2$. By comparing series expansions of the form (3) it may be possible to show even a stronger result that the inequalities are valid term by term, i.e.

$$\frac{(u_1 + \cdots + u_n)^m}{n^m} \le \frac{P(n,m)}{C(n+m-1,m)} \le \frac{u_1^m + \cdots + u_n^m}{n} \tag{18}$$

for $u_i \ge 0$, $i = 1, 2, \ldots, n$. Then (17) is also valid when any of the u's is < 0, i.e. any of the x's $\in (0,1)$, since for any of these means, say M, we have $M(ax_1, \ldots, ax_n) = aM(x_1, \ldots, x_n)$ for all $a > 0$.

The `LOGMEAN` program includes options for checking the validity of (17) and (18). In rather extensive numerical tests no violation against these conjectures have been found.

10. Appendix 1: Proof of (18) in case $n = 2$ (26 December 2002)

When $n = 2$ it is sufficient to study the case $u_1 = u$, $u_2 = 1$ and assume that $u > 1$. Then (18) can be written as

$$\frac{(u+1)^m}{2^m} \le \frac{u^{m+1} - 1}{(m+1)(u-1)} \le \frac{u^m + 1}{2} \tag{19}$$

The second part of this double inequality is equivalent to

$$2(u^{m+1} - 1) \le (m+1)(u-1)(u^m + 1)$$

or

$$f(u) = (m-1)u^{m+1} - (m+1)u^m + (m+1)u - (m-1) \geq 0. \tag{20}$$

By studying the first and second derivatives of $f(u)$ it can be easily seen that (20) holds.

The first part of the double inequality is equivalent to

$$(m+1)(u-1)(u+1)^m \leq 2^m(u^{m+1}-1)$$

or

$$g(u) = 2^m(u^{m+1}-1) - (m+1)(u-1)(u+1)^m \geq 0. \tag{21}$$

It can be shown by induction that the k^{th} derivative of $g(u)$ is

$$g^{(k)}(u) = \frac{(m+1)!}{(m-k+1)!}[2^m u^{m-k+1} - k(u+1)^{m-k+1} - (m-k+1)(u-1)(u+1)^{m-k}]$$

for $k \leq m+1$ and $g^{(k)}(u) = 0$ for $k > m+1$. Especially when $u = 1$ we have

$$g^{(k)}(1) = \frac{(m+1)!}{(m-k+1)!}2^{m-k+1}(2^{k-1}-k),\ k \leq m+1.$$

Thus $g(u)$ and all its derivatives are non-negative for $u = 1$ and from the Taylor expansion of $g(u)$ we can deduce that (21) holds for all m.

11. Appendix 2: Proof of the first part of (18) (10 June 2003) by Jorma Merikoski

Let $u_1, ..., u_n \geq 0$. Their m'th "symmetric mean" (see e.g. Mitrinović [5], p. 95) is defined by

$$s_m(u_1, ..., u_n) = C(n,m)^{-1} \sum_{1 \leq i_1 < ... < i_m \leq n} u_{i_1} ... u_{i_m}.$$

Allowing also equal i_k's, we meet the "generalized m'th symmetric mean" (see e.g. [5], p. 105, note that $C(n+m-1, m) = C(n+m-1, n-1)$), defined by

$$h_m(u_1, ..., u_n) = C(n+m-1, m)^{-1} \sum_{i_1 + ... + i_n = m} {u_1}^{i_1} ... {u_n}^{i_n} \quad (i_1, ..., i_n \geq 0),$$

which appears in the middle of (18). (Here we define $0^0 = 1$. In fact, the functions s_m and h_m should not be called means, since they are not homogeneus and all their values are not between $\min_i u_i$ and $\max_i u_i$. Neither should h_m be called a generalization of s_m, since s_m is not obtained from h_m as a special case. The functions $s_m^{1/m}$ and $h_m^{1/m}$ are actual means.)

Fix $u_1, ..., u_n$. Neuman ([8], Corollary 3.2) proved that

$$k \leq m \Rightarrow h_k^{1/k} \leq h_m^{1/m}. \tag{22}$$

Putting $k = 1$ proves the first part of (18). The second part remains open.

DeTemple and Robertson [2] gave an elementary proof of (22) for $n = 2$, but Neuman's proof for general n is not elementary, applying B-splines. The problem, whether the first part of (18) has an elementary proof, and the stronger problem, whether (22) has such a proof, remain also open.

12. Appendix 3: Alternative derivations of (4). Proofs of (17) (7 October 2003) by Jorma Merikoski

I noted only recently that alternative derivations of (4) and proofs of (17) appear in the literature.

Neuman [9] defined (as a special case of [9], Eq. (2.3))

$$L(x_1, ..., x_n) = \int_{E_{n-1}} \Big(\exp \sum_{i=1}^{n} v_i \log x_i \Big) dv, \tag{23}$$

where $v_1 + ... + v_n = 1$,

$$E_{n-1} = \{(v_1, ..., v_{n-1}) \,|\, v_1, ..., v_{n-1} \geq 0,\ v_1 + ... + v_{n-1} \leq 1\},$$

and $dv = dv_1 ... dv_{n-1}$. He ([9], Theorem 1 and the last formula) proved (17) and reduced (23) into (4).

Pečarić and Šimić [10] tied Neuman's approach to a wider context. They studied extensively various logarithmic and other means. As a special case ([10], Remark 5.4), they obtained (4).

Xiao and Zhang (unaware of [9]) defined

$$L(x_1, ..., x_n) = \frac{(n-1)!}{V(\log x_1, ..., \log x_n)} \sum_{i=1}^{n} (-1)^{n+i} x_i V_i(\log x_1, ..., \log x_n), \tag{24}$$

where V denotes the Vandermonde determinant and V_i is obtained from it by omitting the last row and i'th column. Actually (24) equals (4). Also they proved (17).

The current version of this paper can be downloaded from
`http://www.survo.fi/papers/logmean.pdf`

13. Appendix 4: An update (17 November 2005) by Jorma Merikoski

Motivated by this paper, I [J. Ineq. Pure Appl. Math. 5 (2004), Article 65] surveyed and further developed its results. Neuman [SIAM J. Math. Anal. 19 (1988), 736-750] proved the second part of (18).

References

[1] B.C.Carlson, *The logarithmic mean*, Amer. Math. Monthly, 79 (1972), 615–618.

[2] D.W.DeTemple and J.M.Robertson, *On generalized symmetric means of two variables*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 634-677 (1979), 236–238.

[3] E.L.Dodd, *Some generalizations of the logarithmic mean and of similar means of two variates which become indeterminate when the two variates are equal*, Ann. Math. Stat., 12 (1941), 422–428.

[4] C.-E.Fröberg, *Introduction to numerical analysis*, Addison-Wesley, 1965.

[5] D.S.Mitrinović, *Analytic Inequalities*, Springer, 1970.

[6] S.Mustonen, *A generalized logarithmic mean*, unpublished manuscript, University of Helsinki, Dept. of Statistics, 1976.

[7] S.Mustonen, *Survo - An integrated environment for statistical computing and related areas*, Survo Systems, 1992.

[8] E.Neuman, *Inequalities involving generalized symmetric means*, J. Math. Anal. Appl. 120 (1986), 315–320.
[9] E.Neuman, *The weighted logarithmic mean*, J. Math. Anal. Appl. 188 (1994), 885-900.
[10] J.Pečarić and V.Šimić, *Stolarsky-Tobey mean in n variables*, Math. Ineq. Appl. 2 (1999), 325-341.
[11] A.O.Pittenger, *The logarithmic mean in n variables*, Amer. Math. Monthly, 92 (1985), 99–104.
[12] L.Törnqvist, *A memorandum concerning the calculation of Bank of Finland consumption price index*, unpublished memo, Bank of Finland, Helsinki, 1935 (Swedish).
[13] L.Törnqvist, P.Vartia and Y.O.Vartia, *How should relative changes be measured?*, Amer. Statistician, 39 (1985), 43–46.
[14] Y.O.Vartia, *Ideal log-change index numbers*, Scand. J. of Statistics, 3 (1976), 121–126.
[15] Z-G.Xiao and Z-H.Zhang, *The inequalities $G \leq L \leq I \leq A$ in n variables*, J. Ineq. Pure Appl. Math. 4 (2003), Article 39.

Department of Statistics, University of Helsinki
E-mail address: seppo.mustonen@helsinki.fi