ENUMERATION
OF
UNIQUELY SOLVABLE OPEN SURVO PUZZLES

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1. Introduction

In an $m \times n$ Survo puzzle an $m \times n$ table has to be filled by integers $1, 2, \ldots, mn$ in such a way that the their row and column sums are equal to integers given as marginals. Often some of the numbers $1, 2, \ldots, mn$ are placed readily in the table in order to guarantee that the solution is unique or to make the problem easier. A Survo puzzle is called open if the marginals only are given. Finding all open, uniquely solvable Survo puzzles is not a trivial task.
Statistically speaking, it may seem surprising that such tables exist. An open Survo puzzle can be seen as a contingency table with row and column totals given but all cell frequencies missing. Usually the totals or marginal sums tell practically nothing about the cell frequencies. However, the restricition that the cell frequencies are exactly the integers $1, 2, \ldots, mn$ in certain order is so strong that sometimes the marginals determine the cell frequencies uniquely.

2. Example

Consider the following 3×4 contingency table with missing cell frequencies:

	1	2	3	4	
1					13
2					27
3					38
	13	14	23	28	78

In this case there are a huge amount (1104682) of different tables having the same margins, two extreme examples being

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	1	2	3	4	
1	0	0	0	13	13
2	0	0	12	15	27
3	13	14	11	0	38
	13	14	23	28	78

	1	2	3	4	
1	13	0	0	0	13
2	0	14	13	0	27
3	0	0	10	28	38
	13	14	23	28	78

with correlation coefficients $r = -0.78$ and $r = 0.92$, respectively, when equally spaced scores are assumed.

The grand total in these tables is 78 which happens to be equal to $1+2+3+4+5+6+7+8+9+10+11+12$. The only case among these over a million cases containing just these cell frequencies is

	1	2	3	4	
1	2	1	3	7	13
2	5	4	8	10	27
3	6	9	12	11	38
	13	14	23	28	78

There is no direct proof that there are no other solutions for this open Survo puzzle. The only way to show the uniqueness is to use an algorithm which sorts out all possible solutions.

On the other hand, it is easy to create, for example, 3×4 tables with cell counts $1, 2, \ldots, 12$ in some order. There are altogether $12! = 479001600$ such tables but only 83952 (0.0175 per cent) of them presented as open Survo puzzles (only marginals given) have a unique solution. Of these 83952 tables only $83952/(3! \cdot 4!) = 583$ are essentially different when tables obtained from each other by row and column permutations are considered similar.

3. Enumeration

Two open, uniquely solvable Survo puzzles A and B are defined essentially different if the solution of A cannot be transformed into the solution of B by interchanging rows and columns or by transposing (in case $m = n$). Let $S(m, n)$ be the number of essentially different open $m \times n$ Survo puzzles. It is evident that $S(1, 1) = 1$ and $S(2, 2) = 1$, see [2].

Currently (October 2007) the following $S(m,n)$ numbers have been found:

m/n	2	3	4	5	6	7	8	9	10
2	1	18	62	278	1146	5706	28707	154587	843476
3	18	38	583	5337	55815	617658			
4	62	583	5327	257773					
5	278	5337	257773						
6	1146	55815							
7	5706	617658							
8	28707								
9	154587								
10	843476								

$S(3,3) = 38$ was determined by *Reijo Sund* who was the first to pay attention to enumeration of open Survo puzzles. He calculated this result by studying all $9! = 362880$ possible 3×3 tables by the standard combinatorial and data handling program modules of SURVO MM (see Appendix 1 in [2]).

A much more efficient way for determining $S(m,n)$ is to start from all possible partitions of margins, i.e. from possible distinct partitions of the total sum $S = mn(mn+1)/2$ to sums of m and n integers. If the number of distinct partitions of r into s parts with minimal part min and maximal part max is denoted by $P(r,s,min,max)$, there are when $m \neq n$

$$P(S,m,n(n+1)/2,mn^2-n(n-1)/2)P(S,n,m(m+1)/2,m^2n-m(m-1)/2)$$

tables to be scanned for uniqueness of the solution and this is much less than $(mn)!$.

For example, when $m = 3$ and $n = 4$, the numbers of those partitions partitions are $P(78,3,10,42) = 128$ and $P(78,4,6,33) = 519$ obtained in Survo by the `COMB` commands

```
____________________________________________________________________
OFF=2,3,14,4,12,8,6,13,5,9,25,11
The sum of all numbers in a 3x4-table is 12*13/2=78.
COMB P,CUR+1 / P=PARTITIONS,78,3 DISTINCT=1 MIN=10 MAX=42 RESULTS=0
Partitions 3 of 78: N[P]=128
COMB P,CUR+1 / P=PARTITIONS,78,4 DISTINCT=1 MIN=6 MAX=33 RESULTS=0
Partitions 4 of 78: N[P]=519
____________________________________________________________________
```

and thus there are only $128 \cdot 519 = 66432$ cases to be scanned instead of $12! = 479001600$.

The number $S(3,4) = 583$ was computed by my original solver program (see [2], page 11). Already the number $S(4,4)$ would have been absolutely formidable to compute by my original program, but *Petteri Kaski* could do that rather easily by converting the task into an exact cover problem and got the result $S(4,4) = 5327$.

The most efficient solver program for open Survo puzzles seems to be `SP_SOL` which I created during Summer 2007 and the current table of known $S(m,n)$ values is computed by it. The new program confirms the earlier results.

4. Main features of the new solver program

The idea of this program is simple, but its efficiency is essentially dependent on how various stages of the solution are implemented.

The working principle is illustrated here by the next example given as the Problem 11 in [1], page 24. The row and column sums are here reversed to ascending order and thus the Survo puzzle

				17
				32
				36
				51
17	26	42	51	

is to be considered. The only solution is

2	1	5	9	17
3	6	10	13	32
4	7	11	14	36
8	12	16	15	51
17	26	42	51	

At first, no attention is paid on column sums. The target is to find preliminary solutions giving correct row sums 17,32,36,51.

The procedure starts from the first row and the `COMB` program (used here only as an demonstration vehicle) gives the following 11 restricted partitions:

```
__________________________________________________________________
COMB P,CUR+1 / P=PARTITIONS,17,4 DISTINCT=1 MAX=16
Partitions 4 of 17: N[P]=11
1 2 3 11
1 2 4 10
1 2 5 9    !
1 2 6 8
1 3 4 9
1 3 5 8
1 3 6 7
1 4 5 7
```

```
2 3 4 8
2 3 5 7
2 4 5 6
____________________________________________________________________
```

From these partitions each in turn should be investigated but here a shortcut is taken by selecting the only one leading to a genuine solution, namely the third (!) 1 2 5 9.

Next, the all alternatives for the row 2 are sought for. These partitions of 32 must not contain numbers 1,2,5,9 and they are found as follows:

```
____________________________________________________________________
OFF=1,2,5,9
COMB P,CUR+1 / P=PARTITIONS,32,4 DISTINCT=1 MAX=16
Partitions 4 of 32: N[P]=16
3 4 10 15
3 4 11 14
3 4 12 13
3 6 7 16
3 6 8 15
3 6 10 13   !
3 6 11 12
3 7 8 14
3 7 10 12
3 8 10 11
4 6 7 15
4 6 8 14
4 6 10 12
4 7 8 13
4 7 10 11
6 7 8 11
____________________________________________________________________
```

Again, each of these 16 partitions must be studied separately, but as in the first stage here, the only favourable one, the sixth (!) 3 6 10 11 is selected.

Similarly, the candidates for the row 3 are found by observing what has selected already on the previous rows by

```
____________________________________________________________________
OFF=1,2,5,9,3,6,10,13
COMB P,CUR+1 / P=PARTITIONS,36,4 DISTINCT=1 MAX=16
Partitions 4 of 36: N[P]=1
4 7 11 14
____________________________________________________________________
```

and this leads to only one possibility. The last row then must contain the remaining numbers. In this case they are 8,12,15,16.

Thus a preliminary solution with correct row sums

```
   A  B  C  D
1  1  2  5  9  17
2  3  6 10 13  32
3  4  7 11 14  36
4  8 12 15 16  51
  16 27 41 52       computed sums
  17 26 42 51       correct sums
  -1  1 -1  1       error
```

is then found, but the column sums taken now into consideration do not match. The final solution(s), if any, must be found by swapping numbers within the rows. In this case it is easy to detect that the swaps (1,2),(15,16) and these only lead to the final solution.

In reality, the solution(s) must be found without shortcuts made in the previous instance. The solver program has to scan all possible alternatives. In this case a thorough search leads to 394 preliminary solutions with correct row sums, one of them being

```
   A  B  C  D
1  1  2  3 11  17
2  5  6  9 12  32
3  4  8 10 14  36
4  7 13 15 16  51
  17 29 37 53       computed sums
  17 26 42 51       correct sums
   0  3 -5  2       error
```

Now the first column sum is correct but the second sum is helplessly too large since it cannot be diminished by any swap within the rows (where numbers are always in increasing order). A similar deadlock is encountered by all the other 394 candidates save the only happy one.

The solver program includes two recursive algorithms. The first one finds all candidates satisfying the row sums. The second one studies each candidate in turn and finds all final solutions by swaps within the rows. In both algorithms the essential task is to find restricted integer partitions with different kind of restrictions in each.

5. Reducing sets of marginal partitions

Any solver program for Survo puzzles based on scanning the combinations of marginal partitions can be speeded up by observing that the sets of these partitions may be reduced by imposing simple linear constraints for cumulative sums of marginal sums.

Let us consider an open $m \times n$ Survo puzzle with marginal sums $r_1 < r_2 < ... < r_m$ and $c_1 < c_2 < ... < c_n$. The corresponding cumulative sums are denoted

$$R_i = r_1 + \cdots + r_i, \quad i = 1, 2, \ldots, m,$$
$$C_j = c_1 + \cdots + c_j, \quad j = 1, 2, \ldots, n.$$

It can be easily verified that the puzzle has no solution if at least one of the conditions

$$(1) \qquad R_i < 2in(2in+1)/2, \quad i = 1, 2, ..., m, \\ C_j < 2im(2im+1)/2, \quad j = 1, 2, ..., n.$$

is satisfied.

The number of cases to be studied thoroughly by a solver program is still more substantially reduced by observing that the puzzle has no solutions if at least one of the conditions

$$(2) \quad (jm + in - ij)(jm + in - ij + 1)/2 + ij(ij + 1)/2 < R_i + C_j, \\ i = 1, ..., m-1, \quad j = 1, ..., n-1,$$

is satisfied. Also these conditions are easy to prove.

Instead of a general proof the validity of conditions (2) is shown in the case $i = j = 2$. Then (2) has the form

$$(3) \qquad (2m + 2n - 4)(2m + 2n - 3)/2 + 10 < r_1 + r_2 + c_1 + c_2.$$

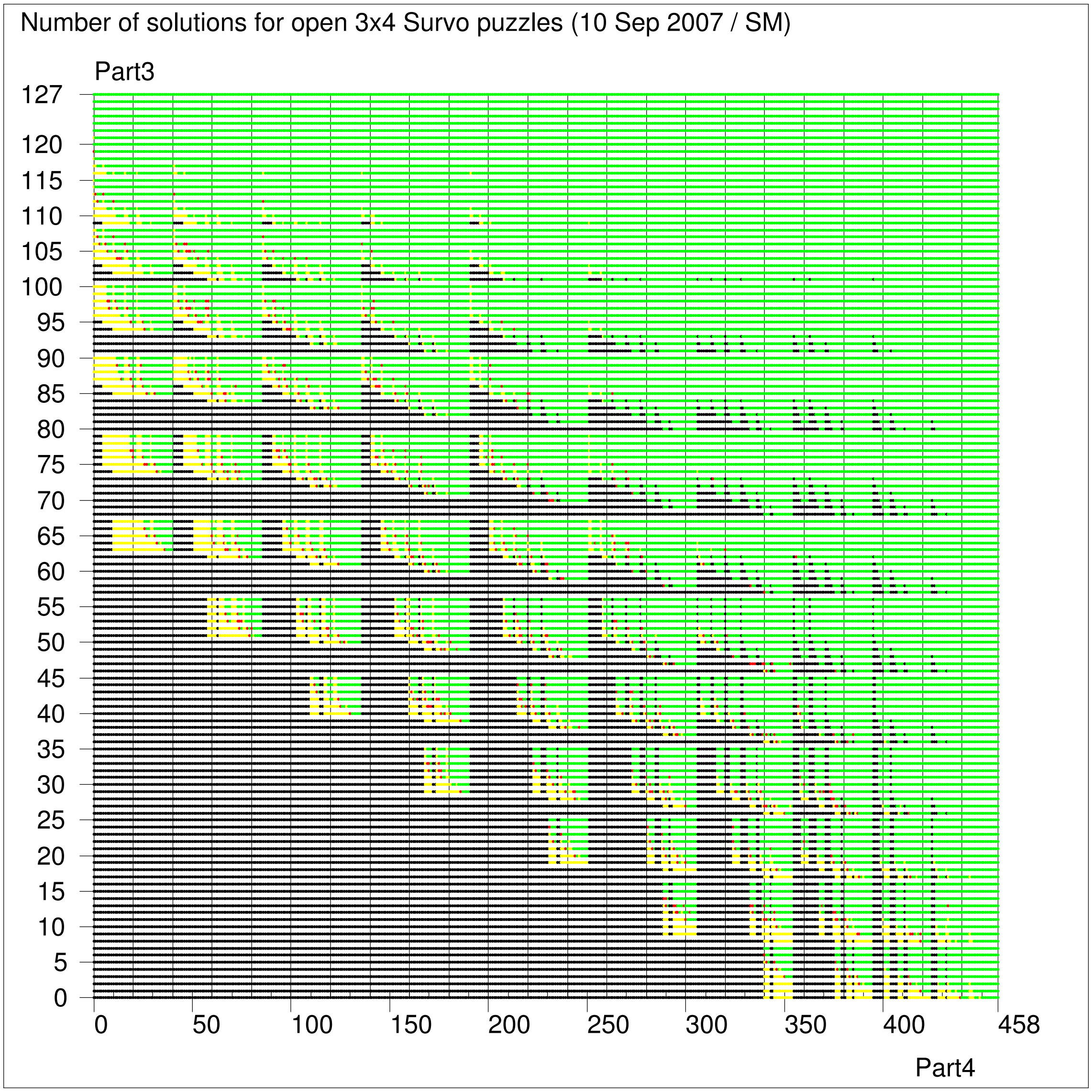
and a possible solution of the puzzle may look like

1	2	·	·	·	r_1
3	4	·	·	·	r_2
·	·	·	·	·	
·	·	·	·	·	
c_1	c_2				

The sum $r_1 + r_2 + c_1 + c_2$ takes its smallest value when in the left upper corner we have the four smallest numbers 1,2,3,4 (in some order) and in the two first rows and in the two first columns we have in other positions the smallest numbers above 4. Therefore in these rows and columns ($2m+2n-4$ positions) the minimal setting consists of numbers $1, 2, \ldots, 2m + 2n - 4$. The total sum of these numbers is $(2m+2n-4)(2m+2n-3)/2$ and because the sum $r_1+r_2+c_1+c_2$ includes the numbers 1,2,3,4 twicely (sum equal to 10) the lower bound for $r_1 + r_2 + c_1 + c_2$ is the left side of (3). A general proof is a straightforward generalization.

Unfortunately, conditions (1) and (2) do not cancel all impossible cases. For example, if the row sum r_1 is minimal and thus it contains the numbers $1, 2, 3, 4, \ldots, n$, the 2×2 left upper corner cannot hold numbers 1,2,3,4 but, at the lowest 1,2,5,6 which makes the condition (3) still stronger. These kind of enhancements have not yet been implemented. The good news are that conditions (1) and (2) cancel a substantial majority of puzzles doomed to lack any solution. For example, in case $m = n = 4$ the original number of cases to be solved is $2980 \cdot 2979/2 = 4438710$ and the number of unsolvable cases is then 1837169 (41.4 %), but when the conditions (1) and (2) are applied, their number drops to 193695 (4.4 %).

A smaller case $m = 3, n = 4$ is suitable even for graphical illustration. The following graph tells how various cases are distributed in a lattice determined the indices of marginal partitions listed in a lexicographic order. The graph is plotted by Survo using e.g. the specification `POINT_COLOR`.



In this picture, each combination of marginal partitions, i.e. each open 3×4 Survo puzzle is represented by a colored dot.

Interpretation of colors:

Color	Cases	
black	22821	0 solutions, according to (2)
red	583	1 solution
green	32505	2 or more solutions
yellow	2843	0 solutions, not detected by (2)
total	58752	

The cases cancelled by conditions (1) (60 partitions of the total sum into 4 distinct parts) are omitted from the graph. Then it covers all remaining $(519-60)\cdot 128 = 58752$ combinations.

The graph gives a general impression about the nature of the problem. It is good to zoom in to a suitable degree in order to see how 'nastily' the red 'gems' are located principally among the green and yellow dots.

The current version of this paper can be downloaded from
`http://www.survo.fi/papers/enum_survo_puzzles.pdf`

References

[1] http://www.survo.fi/english
[2] http://www.survo.fi/papers/puzzles.pdf