ON THE ROOTS OF AN ALGEBRAIC EQUATION RELATED TO REGULAR POLYGONS

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Abstract. Regular n-sided polygons inscribed in a unit circle are studied. As told in [3] the largest root of an algebraic equation (1) is $R(n,1)^2 = L(n)^2$ where $L(n)$ is the total length of edges and chords of that polygon. This was proved in [5].

Thus $r(n,1) = R(n,1)/n$ is a linear combination of chord lengths with all coefficients equal to 1. When n is even, the largest chord, diameter is halved.

Let the other roots be $R(n,i)^2, i = 2,\ldots,n$. In [3] it was made evident that also $r(n,i) = R(n,i)/n, i = 2,\ldots,n$ are linear combinations of chord lengths with coefficients -1,0,1. When n is a prime or a power of 2, all coefficients are 1 or -1.

Now an essentially faster algorithm for calculating these linear combinations is described. This became possible after finding an efficient computational solution for the riddle of the q coefficients encountered in [3] (pp. 31-33). Before presenting these new results, the main content of [3] will be repeated.

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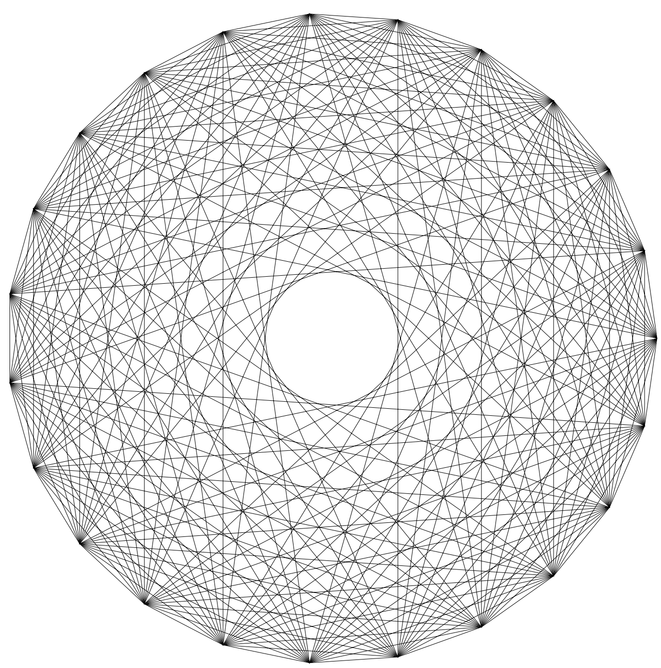

Figure 1. Regular 23-gon with all diagonals

1. Older findings

As described in [3] (pp. 7-9) I found the equation

$$\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n}{2i+k} n^{n-2i-k} x^i = 0 \tag{1}$$

(where $k = 0$ when n is even and $k = 1$ when n is odd)
giving $L(n)^2$ as the largest root, originally by making computational experiments with Survo and Mathematica. This description went as follows:

```
______________________________________________________________________
*Let's start by studying a heptagon (n=7).
*Calculating the square of the total sum of chords with a high accuracy
*(1000) and finding the most plausible equation:
*
*SAVEP CUR+1,E,K.TXT
*n=7;
*a=N[n*Sum[2*Sin[i*Pi/n],{i,1,(n-1)/2}],1000];
*InputForm[RootApproximant[a^2]]
E
*/MATH K.TXT
*In[2]:= n=7;
*In[3]:= a=N[n*Sum[2*Sin[i*Pi/n],{i,1,(n-1)/2}],1000];
*In[4]:= InputForm[RootApproximant[a^2]]
*Out[4]//InputForm= Root[-823543 + 84035*#1 - 1029*#1^2 + #1^3 & , 3, 0]
*
*An equation of 3rd degree is found with the following coefficients
*being multiples of decreasing powers of 7 except in the highest term:
*
*Coefficients                 Coefficients of 7^i, i=0,1,...,(n-1)/2
*823543(10:factors)=7^7              -1
*84035(10:factors)=5*7^5              5
*1029(10:factors)=3*7^3              -3
*1                                    1
*
*A corresponding calculation with values n=11,13,17,19,23 completes
*the following table of coefficients divided by n^i, i=n,n-2,n-4,...
*(c refers to the constant term and it can be fixed to +1)
*i
* n  c   x  x^2    x^3   x^4    x^5    x^6    x^7   x^8  x^9 x^10 x^11
* 7  1  -5    3     -1
*11  1 -15   42    -30     5     -1
*13  1 -22   99   -132    55     -6      1
*17  1 -40  364  -1144  1430   -728    140     -8     1
*19  1 -51  612  -2652  4862  -3978   1428   -204     9   -1
*23  1 -77 1463 -10659 35530 -58786  49742 -21318  4389 -385   11    1
*
*The general form of the polynomial is
*P(n,x)=S(n,0)*n^n + S(n,1)*n^(n-2)*x + ... + S(n,k-1)*n^3*x^(k-1) + x^k
```

```
*where k=(n-1)/2.
*
*Temporarily absolute values of S(n,i)'s denoted here by Sni are
*studied.
*The 'law' for Sn1's is revealed by ESTIMATE operation of Survo
*from the data set S1 (corresponding to x column above):
*
*DATA S1
* n   Sn1
* 7     5
*11    15
*13    22
*17    40
*19    51
*23    77
*
*The dependency between Sn1 of n cannot be linear.
*Therefore a quadratic model MS1 is defined
*MODEL MS1
*Sn1=c0+c1*n+c2*n^2
*
*and coefficients c0,c1,c2 estimated by activating the following line:
*ESTIMATE S1,MS1,CUR+1 / RESULTS=0 METHOD=N
*Estimated parameters of model MS1:
*c0=0.333333 (1.02933E-012)
*c1=-0.5 (1.47693E-013)
*c2=0.166667 (4.85874E-015)
*n=6 rss=0.000000 R^2=1.00000 nf=11
*
*It is then obvious that
*Sn1=1/3-n/2+n^2/6 = (2-3*n+n^2)/6 = (n-1)*(n-2)/6
*and the result is easily checked for each value in DATA S1.
*
*On the basis of this result it is natural to try a quartic model
*for Sn2 values (x^2 column above):
*
*DATA S2
* n    Sn2
* 7      3
*11     42
*13     99
*17    364
*19    612
*23   1463
*
*MODEL MS2
*Sn2=c0+c1*n+c2*n^2+c3*n^3+c4*n^4
*
```

```
*ESTIMATE S2,MS2,CUR+1 / RESULTS=0 METHOD=N
*Estimated parameters of model MS2:
*c0=0.2 (1.09461E-005)
*c1=-0.416667 (3.51571E-006)
*c2=0.291667 (3.93926E-007)
*c3=-0.0833333 (1.84408E-008)
*c4=0.00833333 (3.07131E-010)
*n=6 rss=0.000000 R^2=1.00000 nf=22
*
*These results give credence to following deductions:
*Sn2(n):=1/5-5/12*n+7/24*n^2-1/12*n^3+1/120*n^4
*       =(n-1)*(n-2)*(n-3)*(n-4)/fact(5)     fact() is factorial in Survo
*       =fact(n-1)/fact(n-5)/fact(5)
*       =C(n,5)/n
*       =C(n,2*i+1)/n  (i=2 for this Sni)
*
*For example, C(23,2*2+1)/23=1463 = Sn2(23)
*
*Thus the general expression for numbers S(n,i) is
*
*S(n,i)=(-1)^i*C(n,2*i+1)/n, i=0,1,2,...,(n-1)/2-1
*
*and then the coefficients of the polynomial P(n,x) are
*
*(-1)^i*C(n,2*i+1)*n^(n-2*i-1), i=0,1,2,...,(n-1)/2-1
*
--------------------------------------------------------------------
```

According to this experiment, $L(n)^2$ for at least for primes n is a root of equation

$$\sum_{i=0}^{(n-1)/2} (-1)^i \binom{n}{2i+1} n^{n-2i-1} x^i = 0 \tag{2}$$

where the constant term is n^n and the coefficient of highest term is either 1 or -1 depending on whether $(n-1)/2$ is even or odd. In fact all $(n-1)/2$ roots of equation (2) are real and $L(n)^2$ is the greatest root. The equation seems to be valid also for any odd $n \geq 3$.

By simple trials I found that for any even n the corresponding equation follows after small modifications by replacing $(n-1)/2$ by $n/2$ and $2i+1$ (in two places) by $2i$ and then the general equation obviously valid for all $n \geq 3$ reads

$$\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n}{2i+k} n^{n-2i-k} x^i = 0 \tag{3}$$

where $k=0$ when n is even and $k=1$ when n is odd.

Hence, in general, my conjecture is that $L(n)$ is the square root of the greatest root of equation (3). By replacing x by x^2 the equation gives $L(n)$ as its greatest root directly.

2. Older findings: Roots as linear combinations

It was crucial to note[1] that the roots seem to be related to simple linear combinations of the chord lengths

$$e'_i = 2\sin(i\pi/n), \quad i = 1, 2, \ldots, m \tag{4}$$

where $m = \lfloor n/2 \rfloor$ and e'_1 is the edge length. For forthcoming considerations it is better to present them in an opposite order as follows

$$\begin{aligned} e_1 &= e'_m \text{ for odd } n \quad \text{and} \quad e_1 = e'_m/2 = 1 \text{ for even } n, \\ e_i &= e'_{m+1-i}, \quad i = 2, \ldots, m \end{aligned} \tag{5}$$

When n is even, e_1 is the radius instead of the diameter (the longest chord) and then each of the line segments corresponding to lengths (5) appear in the set of all chords exactly n times and the total length $L(n)$ (square root of the largest root of equation (1)) is

$$L(n) = (e_1 + e_2 + \cdots + e_m)n \tag{6}$$

for all $n > 2$.

According to my examinations it turns out that square roots $R_{n,i}$, $i = 1, 2, \ldots, m$ of all roots of equation (1) can be presented in the form

$$R_{n,i} = (c_{n,1}e_1 + c_{n,2}e_2 + \cdots + c_{n,m}e_m)n \tag{7}$$

where coefficients $c_{n,i}$, $i = 1, 2, \ldots, m$ have only values $-1, 0, 1$. For any prime n and power of 2 the only values are -1 and 1.

Denote

$$r_{n,i} = R_{n,i}/n, \quad i = 1, 2, \ldots, m. \tag{8}$$

I had no general formula for the c coefficients, but it was possible to present a simple algorithm for computing them and thus for any given n, exact expressions (as sums of trigonometric terms) for all roots can be found.

When n is a prime, according to this algorithm, the expression for $r_{n,i}$ is found by at most i trials giving correct c values (instead of checking all 2^m possible combinations without an algorithm).

When n is a composite integer, a considerable part of roots are 'inherited' from corresponding setups for factors of n.

For example, when $n = 15$, the setup contain 3 distinct pentagons (and their diagonals) and 5 distinct equilateral triangles but there are also chords (like edges of the 15-sided polygon and other chords) unique to $n = 15$.

The following excerpt from a Survo edit field illustrates the situation numerically. It gives the matrix of the c coefficients and shows how 3 roots of 7 are related polygons with 3 or 5 sides.

```
____________________________________________________________________
Roots in case n=15

Solving equation by Mathematica:
SAVEP CUR+1,CUR+7,K.TXT
n=15;
```

[1] 1 July 2013

```
eq=Sum[(-1)^i*Binomial[n,2*i+1]*n^(n-2*i-1)*x^i,{i,0,(n-1)/2}];
lst=N[Solve[eq == 0, x,Reals],16];
lst2=x/.lst;
lst3=Map[Sqrt,lst2];
lst4=Function[x,x/n]/@lst3;
TableForm[Sort[lst4,Greater]]

r_{15,i} values, i=1,2,...,7:
/MATHRUN K.TXT
Out[8]//TableForm= 9.514364454222585
                   3.077683537175253
                   1.7320508075688773
                   1.1106125148291929
                   0.7265425280053609
                   0.4452286853085362
                   0.2125565616700221

Computing chord lengths e:
n=15 pi=3.141592653589793
MAT E15=ZER((n-1)/2,1)
MAT TRANSFORM E15 BY 2*sin(((n+1)/2-I#)*pi/n)
MAT LOAD E15,12.123456789012345,CUR+2

MATRIX E15
T(E15_by_2*sin(((n+1)/2-I#)*pi/n))
///                     1
  1       1.989043790736547
  2       1.902113032590307
  3       1.732050807568877
  4       1.486289650954788
  5       1.175570504584946
  6       0.813473286151600
  7       0.415823381635519

Coefficients c (found by algorithm):
MATRIX C15
///      1  2  3  4  5  6  7
1        1  1  1  1  1  1  1
r_{5,1}  0  1  0  0  1  0  0
r_{3,1}  0  0  1  0  0  0  0
4        1 -1  1 -1  1 -1  1
r_{5,2}  0  1  0  0 -1  0  0
6       -1  1 -1  1  1 -1  1
7       -1  1  1 -1 -1  1  1

MAT SAVE C15

Checking that C15*E15 gives the r_{15,i} values:
```

```
MAT E15B=C15*E15   / *E15B~C15*E(D15_by_2*sin(((n+1)/2-I#)*pi/n)) 7*1
MAT LOAD E15B
MATRIX E15B
C15*T(E15_by_2*sin(((n+1)/2-I#)*pi/n))
///                     1
  1      9.514364454222585
r_{5,1}  3.077683537175253
r_{3,1}  1.732050807568877
  4      1.110612514829193
r_{5,2}  0.726542528005361
  6      0.445228685308536
  7      0.212556561670022
-------------------------------------------------------------------
```

When the c coefficients in the matrix `C15` are applied to the exact e values (5), the expressions of the exact roots are obtained.

As told in [3] (p.39) I found that the roots of equation (1) can also be expressed as

$$R_{n,i}^2 = [n\cot((2i-1)\pi/(2n))]^2, \quad i = 1, 2, ..., \lfloor n/2 \rfloor. \tag{9}$$

This has been proved in [5].

The algorithm for representing these roots as simple linear combinations of chord lengths will start from reasonable good numerical approximations of $r_{n,i}$ numbers.

The exact roots are then determined in the decreasing order. Especially immediately after the 'trivial' first root, on the step i it is good to know whether the $r_{n,i}$ happens to be $r_{k,1}$ of some factor k of n. Then without solving the corresponding equation it is possible to check this by using a good approximation of $r_{k,1}$.

Such an approximation is obtained by solving n substituted by k from (9) with $i = 1$ giving

$$k = \pi/(2\arctan(n/R_{k,1})). \tag{10}$$

Then if k close enough to a positive integer k_0, that integer must be a divisor of n and the root in question is related to longest chord in a k_0-sided regular polygon.

2.1. **Determining c coefficients in (7).** Cases where n is a prime number are considered for certain specific values. It is shown how the c coefficients are found for $n = 23$ 'in the hard way' by listing all possible (2048) combinations of eleven +1's and -1's.

```
-------------------------------------------------------------------
*SAVE PGON23A  /  Roots of 23-sided regular polygon
*LOAD INDEX
*/LMAX
*                             ACCURACY=16       pi=3.141592653589793
*n=23
*MAT E23=ZER((n-1)/2,1)
*MAT TRANSFORM E23 BY 2*sin(((n+1)/2-I#)*pi/n)
*MAT LOAD E23,12.123456789012345,CUR+2
*
*MATRIX E23
```

```
*T(E23_by_2*sin(((n+1)/2-I#)*pi/n))
*///                      1
*  1       1.995337538381078
*  2       1.958168175364646
*  3       1.884521844237641
*  4       1.775770436804750
*  5       1.633939786020884
*  6       1.461671928556248
*  7       1.262175888652106
*  8       1.039167900070867
*  9       0.796802179692483
* 10       0.539593542314049
* 11       0.272333298192493
*
*..........................................................................
*Computing approximate values of r_23,i, i=1,2,...,11:
*SAVEP CUR+1,E,K.TXT
*n=23 pi=3.141592653589793
*MAT R23=ZER((n-1)/2,1)
*MAT TRANSFORM R23 BY cot((2*I#-1)*pi/(2*n))
*MAT LOAD R23,12.123456789012345,CUR+1
*MATRIX R23
*T(R23_by_cot((2*I#-1)*pi/(2*n)))
*///                      1
*  1      14.619482518287246
*  2       4.812264198989466
*  3       2.813730331357740
*  4       1.929912394084656
*  5       1.416677250256013
*  6       1.070738552066125
*  7       0.813560343762645
*  8       0.608113471288986
*  9       0.434361296238207
* 10       0.280186859974377
* 11       0.137446836347119
*
*..........................................................................
*Creating all possible sets of +1,-1 coefficients:
*
*Integers 1,2,...,2048=2^11 as binary vectors:
*COMB N2 TO K.TXT / N2=INTEGERS,11,2
*
*SHOW K.TXT / Loading lines 301-310 as an example
*0 0 0 0 1 1 0 0 1 0 0
*0 0 0 0 1 1 0 0 1 0 1
*0 0 0 0 1 1 0 0 1 1 0
*0 0 0 0 1 1 0 0 1 1 1
*0 0 0 0 1 1 0 1 0 0 0
```

```
*0 0 0 0 1 1 0 1 0 0 1
*0 0 0 0 1 1 0 1 0 1 0
*0 0 0 0 1 1 0 1 0 1 1
*0 0 0 0 1 1 0 1 1 0 0
*0 0 0 0 1 1 0 1 1 0 1
*
*Conversion to a matrix B of all +1,-1 combinations:
*FILE SAVE K.TXT TO NEW B / FIRST=1
*MAT SAVE DATA B TO B
*MAT TRANSFORM B BY 2*X#-1
*
*MAT LOAD B(301:310,*),12,CUR+1
*MATRIX B
*T(B_by_2*X#-1)
*///      X1 X2 X3 X4 X5 X6 X7 X8 X9 X1 X1
*  301    -1 -1  1 -1 -1  1 -1  1  1 -1 -1
*  302    -1 -1  1 -1 -1  1 -1  1  1 -1  1
*  303    -1 -1  1 -1 -1  1 -1  1  1  1 -1
*  304    -1 -1  1 -1 -1  1 -1  1  1  1  1
*  305    -1 -1  1 -1 -1  1  1 -1 -1 -1 -1
*  306    -1 -1  1 -1 -1  1  1 -1 -1 -1  1
*  307    -1 -1  1 -1 -1  1  1 -1 -1  1 -1
*  308    -1 -1  1 -1 -1  1  1 -1 -1  1  1
*  309    -1 -1  1 -1 -1  1  1 -1  1 -1 -1
*  310    -1 -1  1 -1 -1  1  1 -1  1 -1  1
*
*Computing all 2048 possible linear combinations with these coefficients:
*MAT A=B*E23        / *A~T(B_by_2*X#-1)*T(E23_by_2*sin(((n+1)/2-I#)*pi/n)) 2048*1
*
*List of r_{23,i} values and their indices in matrix A:
*
* i   r_{23,i}              index
*
* 1   14.619482518287245     2048
* 2   4.812264198989465      1463
* 3   2.813730331357741       925
* 4   1.9299123940846557      871
* 5   1.4166772502560133      497
* 6   1.0707385520661250     1366
* 7   0.8135603437626450     1593
* 8   0.6081134712889860     1921
* 9   0.4343612962382070      694
*10   0.2801868599743765     1326
*11   0.13744683634711928     820
*
*Searching for the index of any particular r value from matrix A
*loaded below in the edit field:
*FIND 0.137446836
```

```
*
*Loading coefficients for current r:
*MAT LOAD B(820,*),123,CUR+1
*MATRIX B
*T(B_by_2*X#-1)
*///       X1  X2  X3  X4  X5  X6  X7  X8  X9 X10 X11
*  820     -1   1   1  -1  -1   1   1  -1  -1   1   1
*
*All values of linear combinations listed in the current edit field:
*MAT LOAD A,123.1234567890,CUR+1
*MATRIX A
*T(B_by_2*X#-1)*T(E23_by_2*sin(((n+1)/2-I#)*pi/n))
*///                  1
*    1    -14.6194825183
*    2    -14.0748159219
*    3    -13.5402954337
*    4    -12.9956288373
*    5    -13.0258781589
*    6    -12.4812115625
* ....    ..............
*
*Creating matrix C23 of coefficients:
*
*n=23 m=(n-1)/2
*MAT C23=ZER(m,m)
*MAT C23(1,1)=B(2048,*)
*MAT C23(2,1)=B(1463,*)
*MAT C23(3,1)=B(0925,*)
*MAT C23(4,1)=B(0871,*)
*MAT C23(5,1)=B(0497,*)
*MAT C23(6,1)=B(1366,*)
*MAT C23(7,1)=B(1593,*)
*MAT C23(8,1)=B(1921,*)
*MAT C23(9,1)=B(0694,*)
*MAT C23(10,1)=B(1326,*)
*MAT C23(11,1)=B(0820,*)
*
*
*MAT LOAD C23,12,CUR+1
*MATRIX C23
*0&B(2048,*)&B(1463,*)&B(0925,*)&B(0871,*)&B(0497,*)&B(1366,*)&B(1593,*)&B(1921,*)&B(0
*///       1  2  3  4  5  6  7  8  9 10 11
*  1       1  1  1  1  1  1  1  1  1  1  1
*  2       1 -1  1  1 -1  1  1 -1  1  1 -1
*  3      -1  1  1  1 -1 -1  1  1  1 -1 -1
*  4      -1  1  1 -1  1  1 -1 -1  1  1 -1
*  5      -1 -1  1  1  1  1  1 -1 -1 -1 -1
*  6       1 -1  1 -1  1 -1  1 -1  1 -1  1
```

```
*  7       1  1 -1 -1 -1  1  1  1 -1 -1 -1
*  8       1  1  1  1 -1 -1 -1 -1 -1 -1 -1
*  9      -1  1 -1  1 -1  1  1 -1  1 -1  1
* 10       1 -1  1 -1 -1  1 -1  1  1 -1  1
* 11      -1  1  1 -1 -1  1  1 -1 -1  1  1
*
-------------------------------------------------------------------
```

It was important to notice certain regularity at least on the first rows of the matrix. The coefficients are *periodical*. The period length on the row i is $2i-1$ and those periods are indicated in red. This was also the reason for presenting the chord lengths in decreasing order.

For more revealing information, a similar computing and search process was completed for $n = 43$ leading to selection of $(43-1)/2 = 21$ linear combinations from $2^{21} = 2097152$ alternatives. It gave the following matrix of coefficients:

```
-------------------------------------------------------------------
Coefficient matrix C43 for n=43:

///      1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21
  1   *  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1
  2   *  1 -1  1  1 -1  1  1 -1  1  1 -1  1  1 -1  1  1 -1  1  1 -1  1
  3   * -1  1  1  1 -1 -1  1  1  1 -1 -1  1  1  1 -1 -1  1  1  1 -1 -1
  4      1 -1  1 -1  1 -1  1  1 -1  1 -1  1 -1  1  1 -1  1 -1  1 -1  1
  5      1 -1 -1  1  1  1 -1 -1  1  1 -1 -1  1  1  1 -1 -1  1  1 -1 -1
  6   *  1 -1  1 -1  1 -1  1 -1  1 -1  1  1 -1  1 -1  1 -1  1 -1  1 -1
  7     -1  1 -1  1  1 -1  1 -1  1  1 -1  1 -1 -1  1 -1  1  1 -1  1 -1
  8     -1  1  1 -1 -1  1  1 -1  1  1 -1 -1  1  1 -1 -1  1  1 -1 -1  1
  9     -1 -1 -1 -1  1  1  1  1  1  1  1  1  1 -1 -1 -1 -1 -1 -1 -1 -1
 10     -1 -1  1  1  1  1  1 -1 -1 -1 -1 -1  1  1  1  1  1 -1 -1 -1 -1
 11      1 -1  1 -1  1 -1  1 -1  1 -1  1 -1  1 -1  1 -1  1 -1  1 -1  1
 12     -1  1  1  1 -1 -1 -1  1  1  1 -1 -1 -1  1  1  1 -1 -1 -1  1  1
 13     -1  1 -1  1  1 -1  1  1 -1  1 -1 -1  1 -1 -1  1 -1  1  1 -1  1
 14      1 -1  1 -1  1  1 -1  1 -1  1 -1 -1  1 -1  1 -1 -1  1 -1  1 -1
 15     -1 -1 -1 -1 -1 -1 -1  1  1  1  1  1  1  1  1  1  1  1  1  1  1
 16      1 -1 -1  1 -1 -1  1  1 -1  1  1 -1 -1  1  1 -1  1  1 -1 -1  1
 17     -1 -1  1  1  1 -1 -1 -1  1  1  1  1 -1 -1 -1  1  1  1 -1 -1 -1
 18      1 -1  1  1 -1 -1  1 -1 -1  1 -1 -1  1  1 -1  1  1 -1  1  1 -1
 19     -1 -1 -1  1  1  1  1  1  1 -1 -1 -1 -1 -1 -1  1  1  1  1  1  1
 20      1  1 -1 -1 -1 -1  1  1  1  1 -1 -1 -1 -1  1  1  1  1 -1 -1 -1
 21      1 -1 -1  1  1 -1 -1  1  1 -1 -1  1  1 -1 -1  1  1 -1 -1  1  1
-------------------------------------------------------------------
```

A similar periodicity prevails here, but the actual coefficients on a given line are not usually the same. When comparing this to to the case $n = 23$, rows denoted by an asterisk have the same pattern, others not.

There is a strong temptation to look for simple trigonometric functions and after some experiments (by plotting trigonometric curves and observing their sign changes)

I came to a conclusion that for primes n the general element of the $C(n)$ matrix is of the form

$$(11)\qquad C(n)_{ij} = \pm\,\mathrm{sgn}(\cos(q_{n,i}\pi(2j-1)/(2i-1))), \quad i,j = 1,2,\ldots,\lfloor n/2\rfloor$$

where coeficients $q_{n,i}$ are positive integers less than i for $i > 1$ and equal to 1 for $i = 1$. The sign of the expression is selected so that the corresponding linear combination gets a positive value.

For example, for $n = 23$ these coefficients are found by means of Survo as follows:

```
___________________________________________________________________
pi=3.141592653589793
n=23
q=5  I=11    0<q<=I 0<J#<=I#
MAT H=ZER(1,(n-1)/2)
MAT #TRANSFORM H BY sgn(cos(q*pi*(2*J#-1)/(2*I-1)))
MAT G!=H*D23
MAT LOAD G,123.123456789012345,CUR+2

MATRIX G
///                     1
  1      -0.137446836347119


 i   r_{23,i}                q_{23,i}   +-

 1   14.619482518287245         1        -
 2   4.812264198989465          1        +
 3   2.813730331357741          1        -
 4   1.9299123940846557         1        -
 5   1.4166772502560133         1        -
 6   1.0707385520661250         5        +
 7   0.8135603437626450         2        +
 8   0.6081134712889860         1        +
 9   0.4343612962382070         7        -
10   0.2801868599743765         7        +
11   0.13744683634711928        5        -
___________________________________________________________________
```

In the above display (line 3) the combination `q=1 I=1` gives always $r_{n,1}$ and for other rows the right `q` value is found by a systematic search starting from `q=1`.

As mentioned earlier, for composite n some of the linear combinations are inherited from corresponding calculations of some factors of n. In such a case, no valid `q` coefficient is found according to (11) and then the correct factor is found by using (10) for $r = r_{n,i}$.

When n is even, the formula (11) is replaced by

$$(12)\qquad C(n)_{ij} = \pm\,\mathrm{sgn}(\cos(q_{n,i}\pi(2j-2)/(2i-1))), \quad i,j = 1,2,\ldots,n/2.$$

The structure of $r_{i,n}$ numbers for $n = 30$ is following:

```
-------------------------------------------------------------------
n=30:  pi=3.141592653589793

The roots related to factors of n=30 are revealed by the equation
    k(r):=pi/(2*arctan(1/r))
so that for the second root 6.313751514675043
k(6.313751514675043)=10 refers to r_{10,1}
Thereafter every third root (3*10=30) is inherited from decagon.

Similarly for the third root 3.732050807568877
k(3.732050807568877)=6 refers to r_{6,1}
Thereafter every fifth (5*6=30) root is inherited from hexagon.

                                     q     +-
 1  19.081136687728211               1     +
 2  6.313751514675043     r_{10,1}
 3  3.732050807568877     r_{6,1}
 4  2.605089064693802                2     -
 5  1.9626105055051506    r_{10,2}
 6  1.5398649638145829               2     -
 7  1.2348971565350514               5     +
 8  1.0000000000000000    r_{10,3}
 9  0.8097840331950071               2     +
10  0.6494075931975106               6     -
11  0.5095254494944288    r_{10,4}
12  0.3838640350354158               5     -
13  0.2679491924311227    r_{6,3}
14  0.15838444032453629   r_{10,5}
15  0.05240777928304120             14     +
-------------------------------------------------------------------
```

Seven of the roots are those of either a decagon or a hexagon. The remaining eigth roots are unique for $n = 30$. The task of specifying the exact roots is thus partially recursive leading in this example to examination of cases $n = 10$ and $n = 6$.

The middlemost 8th value is equal to 1 meaning that n^2 is a root of equation (1). By inserting this to the equation leads (assuming that n is even) to

$$\sum_{i=0}^{n/2} (-1)^i \binom{n}{2i} = 0 \tag{13}$$

and it is easy to see that this is true only if n is of the form $n = 2(2k + 1)$.

A more general result valid for any even n is that

$$1/r_{n,i} = r_{n,n/2+1-i}, \quad i = 1, 2, \ldots, n/2. \tag{14}$$

For example, in the preceding example for $n = 30$ we have
$1/r_{30,1} \approx 1/19.081136687728211 \approx 0.05240777928304120 \approx r_{30,15}$,
$1/r_{30,2} \approx 1/6.313751514675043 \approx 0.15838444032453629 \approx r_{30,14}$,

etc.

Equations (14) are proved as follows[2]. Assume that x is a root of (1). Then according to (8) and (14) also n^4/x should be a root of the same equation. This is shown simply by replacing x by n^4/x in (1) and detecting that then the original equation reappears after multiplying by $x^{n/2}/n^n$. Thus n^4/x is also a root of (1).

2.2. **About q coefficients.** As said earlier it is evident that coeficients $q_{n,i}$ are positive integers less than i for $i > 1$ and since $i = 1$ refers to the largest root we have $q_{n,1} = 1$ for all n.

According to numerical experiments the q coefficient for the smallest root is $\lfloor n/4 \rfloor$ when n is odd and $\lfloor n/2 - 1 \rfloor$ when n is even.

Numerical examinations show certain patterns in the behaviour of the q coefficients and so also of rows of the $C(n)$ matrices. In particular, by defining

$$\mathrm{amod}(n,k) = \begin{cases} \mathrm{mod}(n,k), & \text{if } k \le n/2, \\ k - \mathrm{mod}(n,k) & \text{otherwise} \end{cases}$$

I have noticed that if for any two primes n_1, n_2 we have $\mathrm{amod}(n_1, 2i-1) = \mathrm{amod}(n_2, 2i-1)$, then $q_{n_1,i} = q_{n_2,i}$ and thus the patterns of coefficients on row i of $C(n_1)$ and $C(n_2)$ matrices are the same. The same seems to be true also for composite n values when $q_{n,i}$ really exists so that the corresponding $r_{n,i}$ is not related to any factor of n.

For example, the similarities of patterns for $n_1 = 23$ and $n_2 = 43$ on rows $2, 3, 6$ (see p. 11) are consequences of relations
$\mathrm{amod}(23, 2\cdot 2-1) = \mathrm{amod}(43, 2\cdot 2-1) = 1$,
$\mathrm{amod}(23, 2\cdot 3-1) = \mathrm{amod}(43, 2\cdot 3-1) = 2$,
$\mathrm{amod}(23, 2\cdot 6-1) = \mathrm{amod}(43, 2\cdot 6-1) = 1$,
but
$\mathrm{amod}(23, 2\cdot 4-1) = 2, \quad \mathrm{amod}(43, 2\cdot 4-1) = 1$,
$\mathrm{amod}(23, 2\cdot 5-1) = 4, \quad \mathrm{amod}(43, 2\cdot 5-1) = 2$.

In the next table [3] the q coefficients related to primes according to their amod values are given for rows $2, 3, \ldots, 22$. The row i in the table is a permutation of integers $1, 2, \ldots, i-1$. The numbers displayed in gray (being the same as column numbers) extend each row i to a permutation, but cannot appear as amod values due to common factors with $2i - 1$.

row/amod	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
2	1																				
3	2	1																			
4	3	2	1																		
5	4	2	3	1																	
6	5	3	2	4	1																
7	6	3	2	5	4	1															
8	7	4	3	2	5	6	1														
9	8	4	3	2	5	7	6	1													

[2] 14 July 2013

[3] The same table extended to row=75: `http://www.survo.fi/papers/Q75.txt`

10	9	5	3	7	2	8	4	6	1												
11	10	5	3	8	2	6	7	4	9	1											
12	11	6	4	3	7	2	5	10	9	8	1										
13	12	6	4	3	5	2	9	11	7	10	8	1									
14	13	7	3	10	8	6	2	5	9	4	11	12	1								
15	14	7	5	11	3	12	2	9	8	13	4	6	10	1							
16	15	8	5	4	3	13	11	2	12	14	7	9	6	10	1						
17	16	8	3	4	10	6	7	2	9	5	11	12	14	13	15	1					
18	17	9	6	13	5	3	7	11	2	10	8	16	4	14	15	12	1				
19	18	9	6	14	11	3	8	7	2	13	5	17	10	4	16	15	12	1			
20	19	10	3	5	4	6	14	17	9	2	16	12	13	7	15	11	8	19	1		
21	20	10	7	5	4	17	3	18	16	2	13	12	11	19	15	9	6	8	14	1	
22	21	11	7	16	13	18	3	8	12	15	2	9	5	20	10	4	19	6	17	14	1

These values apply also for any composite n in those cases where the root is not related to some factor of n.

The permutations appearing in the table presented by cycles are

row	permutation
3	(1,2)
4	(1,3)
5	(1,4)
6	(1,5)(2,3)
7	(1,6)(2,3)
8	(1,7)(2,4)
9	(1,8)(2,4)(6,7)
10	(1,9)(2,5)(4,7)(6,8)
11	(1,10)(2,5)(4,8)
12	(1,11)(2,6)(3,4)(5,7)(8,10)
13	(1,12)(2,6)(3,4)(7,9)(8,11)
14	(1,13)(2,7)(4,10)(5,8)
15	(1,14)(2,7)(3,5)(4,11)(6,12)(8,9)(10,13)
16	(1,15)(2,8)(3,5)(6,13)(7,11)(9,12)(10,14)
17	(1,16)(2,8)(5,10)(13,14)
18	(1,17)(2,9)(3,6)(4,13)(8,11)(12,16)
19	(1,18)(2,9)(3,6)(4,14)(5,11)(7,8)(10,13)(15,16)
20	(1,19)(2,10)(4,5)(7,14)(8,17)(11,16)
21	(1,20)(2,10)(3,7)(4,5)(6,17)(8,18)(9,16)(11,13)(14,19)
22	(1,21)(2,11)(3,7)(4,16)(5,13)(6,18)(9,12)(10,15)(14,20)(17,19)

showing that all these permutations are of order 2 with certain systematic features. However, no complete rule how the permutations arise is not found.

3. Solving riddle of q coefficients

During last four years I have spent some amount of time in trying to reveal the secret of the q coefficients playing an important role in this research. Since in the triangular table of q's the i'th row is a permutation of integers $1, 2, \ldots, i$ there has been a temptation to look for a common feature of these permutations. I have not succeeded in such a direct approach.

However, at the end of April 2017, when making Survo demos about this topic, I found an algorithmic solution to this problem. This solution is demonstrated in
`http://www.survo.fi/demos/index.html#ex115`
It is based on an upward extension of the table after detecting a simple linear recursion formula for the columns of the table.

The $q_{n,i}$ values depend on n only through $m = \mathrm{amod}(n, i)$ values. If the sequence of integers in the column m of the table of q's is denoted by $q(i, m), i = 1, 2, \ldots$, the recursive relation

$$q(i, m) = 2q(i - m, m) - q(i - 2m, m), \quad i = 1, 2, \ldots \tag{15}$$

seems to be generally valid and the table of q's can be extended by using this recursion backwards and readily available permutations in the form

$$q(i, m) = 2q(i + m, m) - q(i + 2m, m), \quad i = 1, 2, \ldots . \tag{16}$$

Thus the table of q coefficients can be extended into the form (when permutations until i=43 are available):

i/m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	*3	1	1	*6	1	1	*9	1	1	*12	1	1	*15	1	1	*18	1	1	*21
3	2	1	1	2	*5	2	1	1	2	*10	2	1	1	2	*15	2	1	1	2	*20	2
4	3	2	1	1	2	3	*7	3	2	1	1	2	3	*14	3	2	1	1	2	3	*21
5	4	2	*3	1	1	*6	2	4	*9	4	2	*12	1	1	*15	2	4	*18	4	2	*21
6	5	3	2	4	1	1	4	2	3	5	*11	5	3	2	4	1	1	4	2	3	5
7	6	3	2	5	4	1	1	4	5	2	3	6	*13	6	3	2	5	4	1	1	4
8	7	4	*3	2	*5	*6	1	1	*9	*10	2	*12	4	7	*15	7	4	18	2	*20	*21
9	8	4	3	2	5	7	6	1	1	6	7	5	2	3	4	8	*17	8	4	3	2
10	9	5	3	7	2	8	4	6	1	1	6	4	8	2	7	3	5	9	*19	9	5
11	10	5	*3	8	2	*6	*7	4	*9	1	1	*12	4	14	*15	2	8	*18	5	10	*21
12	11	6	4	3	7	2	5	10	9	8	1	1	8	9	10	5	2	7	3	4	6
13	12	6	4	3	*5	2	9	11	7	*10	8	1	1	8	*15	7	11	9	2	*20	3
14	13	7	*3	10	8	*6	2	5	*9	4	11	*12	1	1	*15	11	4	*18	5	2	*21
15	14	7	5	11	3	12	2	9	8	13	4	6	10	1	1	10	6	4	13	8	9
16	15	8	5	4	3	13	11	2	12	14	7	9	6	10	1	1	10	6	9	7	14
17	16	8	*3	4	10	*6	7	2	*9	5	*11	*12	14	13	*15	1	1	*18	13	14	*21
18	17	9	6	13	*5	3	*7	11	2	*10	8	16	4	*14	*15	12	1	1	12	*20	*21
19	18	9	6	14	11	3	8	7	2	13	5	17	10	4	16	15	12	1	1	12	15
20	19	10	*3	5	4	*6	14	17	*9	2	16	*12	*13	7	*15	11	8	*18	1	1	21
21	20	10	7	5	4	17	3	18	16	2	13	12	11	19	15	9	6	8	14	1	1
22	21	11	7	16	13	18	3	8	12	15	2	9	5	20	10	4	19	6	17	14	1

It is crucial to see that the row i starting by a specific permutation of numbers $1, 2, \ldots, i-1$ (in red) is followed by the same numbers in reversed order (in green), then followed by one dummy value and thereafter this scheme is repeated 'forever'. Dummies may also appear in permutations (typically as multiples of the 'correct' number) but it is not harmful since they cannot appear as q coefficients. For simplicity, dummies can be replaced by zeros.

It is also essential to notice that this backward calculation creates plain zeros on the row 1 and any column can be continued upwards by a still simpler recursion so that $q(-i, m) = -q(i+1, m), \quad i = 0, 1, 2, \ldots$.

```
------------------------------------------------------------------
For example, for m=4 we have
i       ... -4 -3 -2 -1  0  1  2  3  4  5 ...
q(i,4)  ... -1 -1 -2 -1  0  0  1  2  1  1 ...

Then it is obvious that the table of q's can be generated simply
row by row using the recursive relation.
For example, assume that we have rows down to 5 ready with upward
'mirror' completions for 5 first columns:

 i/m     1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21
-5
-4      -4 -2  0 -1 -1!
-3      -3 -2 -1 -1 -2
-2      -2 -1 -1 -2! 0
-1      -1 -1  0 -1 -1
 0       0  0  0! 0  0
 1       0  0  0  0  0! 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
 2       1  1! 0  1! 1  0  1  1  0  1  1  0  1  1  0  1  1  0  1  1  0
 3       2  1  1! 2  0  2  1  1  2  0  2  1  1  2  0  2  1  1  2  0  2
 4       3! 2! 1  1  2  3  0  3  2  1  1  2  3  0  3  2  1  1  2  3  0
 5       4! 2  0  1  1  0  2  4  0  4  2  0  1  1  0  2  4  0  4  2  0

Then the start of the next row emerges for the 5 first elements
recursively as (!'s after numbers used in recursion)

 6       5  3  2  4  1

giving the permutation and the row is completed by the rule told above:

 6       5  3  2  4  1  1  4  2  3  5  0  5  3  2  4  1  1  4  2  3  5
------------------------------------------------------------------
```

On basis of these findings it was possible to create an essentiallly faster algorithm for computing the $C(n)$ matrices based on a readily calculated large matrix of q coefficients. The Survo demo
`http://www.survo.fi/demos/index.html#ex115`
tells how this algorithm works by starting from a table of zeros with a seed 1 in the position (2,1) and using three simple rules explained above.

This new algorithm is now available in SURVO MM as `MAT #QFIND(n)` operation and computing the table of q numbers is at least ten times faster than before. The `MAT #ARFIND` operation is now replaced by `MAT #QRFIND` operation which works similarly but does the job much faster by using a readily computed large table of q values. This table can be computed once for all $i \leq n$ by `MAT #QFIND(n)`. By using `MAT #QRFIND` I have computed the linear combinations (with coefficients ± 1) for the roots of the equation (1) for all prime numbers n less than 10000. This is shown in the Survo demo
http://www.survo.fi/demos/index.html#ex116 .
At the same time I have checked that coefficients really are either +1 or -1 and linear combinations give the true roots. It has also been verified that each row i in the table of q's gives a permutation of numbers $1, 2, \ldots, i-1$ (when each possible 0 is replaced by the column index m) and each permutation is of order 2 with $i-1$ as its first element and 1 as the last one.

Although the table of q's was computed only once in this experiment, and it takes a few seconds, the entire checking process lasted on my current PC about 15 hours (a lot of matrix manipulations).
The uniqueness of the representation is still harder to validate. So far this has been established numerically for primes up to 79 by a new `CTEST` operation and it took about 100 hours.
Program listings of `MAT #QFIND`, `MAT #QRFIND`, and `CTEST` operations:
http://www.survo.fi/papers/Q-OPER.TXT

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